Finite Branch Solutions to Painlevé VI Around a Fixed Singular Point*

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Dedicated to Professor Kazuo Okamoto on his sixtieth birthday

Abstract

Every finite branch local solution to the sixth Painlevé equation around a fixed singular point is an algebraic branch solution. In particular a global solution is an algebraic solution if and only if it is finitely many-valued globally. The proof of this result relies on algebraic geometry of Painlevé VI, Riemann-Hilbert correspondence, geometry and dynamics on cubic surfaces, resolutions of Kleinian singularities, and power geometry of algebraic differential equations. In the course of the proof we are also able to classify all finite branch solutions up to Bäcklund transformations.

1 Introduction

We are interested in a *finite branch* local solution to the sixth Painlevé equation around a fixed singular point. We show that every such solution is in fact an *algebraic branch* solution (see Definition 1.1 for the terminology). In particular a global solution is an *algebraic solution* if and only if it is *finitely many-valued* globally. Although the problem under study is local in nature, our solution to it relies on an effective combination of some global technologies and some local tools. The former includes the algebraic geometry of the sixth Painlevé equation, Riemann-Hilbert correspondence, geometry and dynamics on cubic surfaces, Kleinian singularities and their minimal resolutions [15, 16, 17, 18, 20], while the latter includes the power geometry of algebraic differential equation [5, 6, 7], which is a method of constructing formal solutions by means of Newton polygons, and the theory of nonlinear differential equations of "regular singular type" [10, 11], which discusses the convergence of formal solutions.

Let us describe our main results in more detail. First we recall that the sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is a Hamiltonian system of nonlinear differential equations

$$\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q}, \tag{1}$$

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with time variable $z \in Z := \mathbb{P}^1 - \{0, 1, \infty\}$ and unknown functions $q = q(z)$ and $p = p(z)$, depending on complex parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in the 4-dimensional affine space

$$\mathcal{K} := \{ \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \}, \tag{2}$$

where the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z-1)H(\kappa) \;=\; (q_0 q_1 q_z)p^2 - \{\kappa_1 q_1 q_z + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_z\} p + \kappa_0(\kappa_0 + \kappa_4) q_z,$$

with $q_\nu := q - \nu$ for $\nu \in \{0, 1, z\}$. Each of the points 0, 1, ∞ is called a *fixed singular point*.

It is well known that equation (1) has the analytic Painlevé property, that is, any meromorphic solution germ at a base point $z \in Z$ can be continued *meromorphically* along any path in Z emanating from z. Thus a solution can branch only around a fixed singular point. We are interested in finite branch solutions around it, by which we mean the following.

Definition 1.1 A *finite branch* solution to equation (1), say, around $z = 0$ is a local solution $(q(z), p(z))$ on a punctured disk $D^\times = D - \{0\}$ centered at $z = 0$ such that its lift $(\tilde{q}(\tilde{z}), \tilde{p}(\tilde{z}))$ along some finite branched covering $\varphi : (\tilde{D}, \tilde{0}) \to (D, 0)$, $\tilde{z} \mapsto z = \tilde{z}^n$ around $z = 0$ is a single-valued meromorphic function on $\tilde{D}^\times = \tilde{D} - \{\tilde{0}\}$. Such a solution is said to be an *algebraic branch* solution if it can be represented by a convergent Puiseux-Laurent expansion

$$q(z) = \sum_{i \gg -\infty} a_i \, z^{i/n}, \qquad p(z) = \sum_{i \gg -\infty} b_i \, z^{i/n}, \tag{3}$$

with $a_i = b_i = 0$ for all sufficiently small $i \ll 0$, namely, if the lift $(\tilde{q}(\tilde{z}), \tilde{p}(\tilde{z}))$ is a single-valued meromorphic function on $\tilde{D}$ with at most pole at the origin $\tilde{z} = \tilde{0}$.

Problem 1.2 Is any finite branch solution to $\mathrm{P_{VI}}(\kappa)$ an algebraic branch solution ?

In this article we settle this problem in the affirmative as is stated in the following.

Theorem 1.3 *Any finite branch solution to Painlevé VI around a fixed singular point is an algebraic branch solution. In particular a global solution is an algebraic solution if and only if it is finitely many-valued globally. These results are valid for all parameters $\kappa \in \mathcal{K}$.*

It is an interesting problem to consider algebraic solutions to Painlevé VI. Many algebraic solutions have been constructed in [1, 2, 3, 8, 13, 14, 24, 25], but a complete classification seems to be outstanding. We hope that Theorem 1.3 will play an important part in discussing this issue. The following remark explains what Theorem 1.3 signifies and why it is remarkable.

Remark 1.4 Logically, according to Definition 1.1, a finite branch solution $(q(z), p(z))$ around $z = 0$ may have a very transcendental singularity at $z = 0$, to the effect that its lift $(\tilde{q}(\tilde{z}), \tilde{p}(\tilde{z}))$ may have infinitely many poles in $\tilde{D}^\times$ accumulating to the origin $\tilde{z} = \tilde{0}$, or even if such an accumulation phenomenon does not occur, it may have an essential singularity at $\tilde{z} = \tilde{0}$. Rather surprisingly, however, Theorem 1.3 excludes the possibility for a finite branch solution to admit such transcendental phenomena. This result becomes more intriguing if we recall that wild behaviors of a generic solution to Painlevé VI have been observed in [9, 12, 22, 31, 32] and examples of solutions with infintely many poles accumulating to $\tilde{z} = 0$ are given in [12, 31]; such a distribution of poles may be expected for a generic solution, though it is not rigorously verified yet to the author's knowledge. Thus we can think that a finite branch solution is quite distinguished from generic solutions, necessarily being an algebraic branch solution.

$$
\begin{array}{ccccc}
I & & II & & III \\
\{\text{all algebraic branch solutions}\} & \overset{i}{\hookrightarrow} & \{\text{all finite branch solutions}\} & \overset{j}{\hookrightarrow} & \{\text{all solutions}\} \\
\rotatebox{90}{$\hookrightarrow$} & & \| & & \\
\{\text{some algebraic branch solutions}\} & \overset{i'}{\hookrightarrow} & \{\text{all finite branch solutions}\} & & \\
I' & & II & &
\end{array}
$$

Figure 1: Main idea for the proof of Theorem 1.3

The main idea for the proof of Theorem 1.3 is presented in Figure 1. We have natural inclusions i and j in the top line of Figure 1 and we wish to show that the injection i is in fact a surjection. Our strategy consists of the "upper bound part" and the "lower bound part".

(1) *Upper bound part*: In this part we investigate the inclusion $j : II \hookrightarrow III$ in Figure 1, considering how the locus of finite branch solutions is included in the moduli space of all solutions. In other words, we make a confinement of the locus II in the entire space III. What we shall really do is not an upper bound estimation of this locus but rather a pinpoint identification of it. This is the main part of the article and we use the algebraic geometry of Painlevé VI, Riemann-Hilbert correspondence, geometry and dynamics on cubic surfaces, and minimal resolutions of Kleinian singularities [15, 16, 17, 18, 19, 20].

(2) *Lower bound part*: In this part we fill in the diagram of Figure 1 by adding the bottom line to the top one. We try to construct as many algebraic branch solutions as possible in order to make the set I' as large as possible. The construction is based on the power geometry technique developed in [5, 6, 7] and the convergence arguments in [10, 11]. We are done if the set I' is large enough to show that the injection $i' : I' \hookrightarrow II$ is in fact a surjection. This does not mean that we verify the equality $I' = II$ directly. (If such a direct approach were feasible, then our problem would not be difficult from the beginning!) Instead, we prove it very indirectly based on the following idea.

(3) *Key trick*: Suppose that a component A of I' injects into a component B of II. If the cardinalities of A and B are finite and the same, then the injection $i' : A \hookrightarrow B$ is in fact a surjection. If A and B are biholomorphic to $\mathbb{C}$ and the injection $i' : A \hookrightarrow B$ is holomorphic, then it must be a surjection because any holomorphic injection $\mathbb{C} \hookrightarrow \mathbb{C}$ is a surjection (use Casorati-Weierstrass or Picard's little theorem). The same argument holds true if $\mathbb{C}$ is replaced by $\mathbb{C}^{\times}$, since any holomorphic injection $\mathbb{C}^{\times} \hookrightarrow \mathbb{C}^{\times}$ is a surjection (lift it to the universal covering $\mathbb{C} \hookrightarrow \mathbb{C}$). These tricks enable us to identify the component $A \subset I'$ with the component $B \subset II$. We show that each component involved is either of the three types mentioned above. Then we make this kind of argument componentwise to get an identification $I' = II$, which leads to the desired coincidence $I' = I = II$.

In view of the way in which Theorem 1.3 is established, the power geometry technique provides us with an efficient method of identifying all finite branch solutions (up to Bäcklund transformations), which have now turned out to be algebraic branch solutions, by determining the leading terms of their Puiseux-Laurent expansions.

In some sense this article is a counterpart of the previous paper [20] where an ergodic study of Painlevé VI is developed (see also the survey [21]). Put $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$. For each

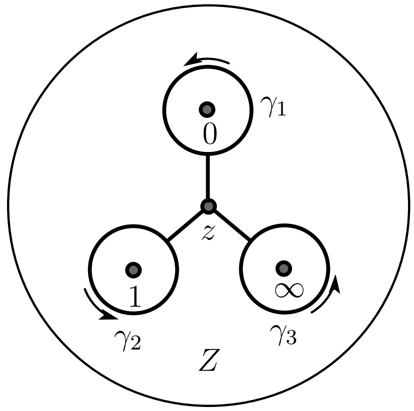


Figure 2: Three basic loops γ_1, γ_2, γ_3 in $Z = \mathbb{P}^1 - \{0, 1, \infty\}$

$\{i, j, k\} = \{1, 2, 3\}$, let γ_i be a loop in Z surrounding z_i once anti-clockwise and leaving z_j and z_k outside as in Figure 2. Then the fundamental group $\pi_1(Z, z)$ is represented as

$$\pi_1(Z, z) = \langle\, \gamma_1, \gamma_2, \gamma_3 \,|\, \gamma_1\gamma_2\gamma_3 = 1 \,\rangle. \tag{4}$$

A loop $\gamma \in \pi_1(Z, z)$ is said to be *elementary* if it is conjugate to γ_i^m for some $i \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$; otherwise, it is said to be *non-elementary*. The main theme of [20] is the dynamics of the nonlinear monodromy of $\mathrm{P_{VI}}(\kappa)$ along a given loop γ. It is shown there that, along every non-elementary loop, the nonlinear monodromy is chaotic and the number of its periodic points grows exponentially as the period tends to infinity. On the other hand, it is Liouville integrable along an elementary loop, in the sense that it preserves a Lagrangian fibration. Now we notice that from the dynamical point of view the main problem of this article is nothing other than discussing the periodic points of the nonlinear monodromy along the basic loop γ_i, which is of course an elementary loop. In view of its integrable character, one may doubt if there is something very deep with this issue. As Theorem 1.3 and Remark 1.4 show, however, this issue is actually quite interesting from the function-theoretical point of view.

The plan of this article is as follows. In §2 the phase space of Painlevé VI is introduced as a moduli space of stable parabolic connections. In §3 the Riemann-Hilbert correspondence from the moduli space to an affine cubic surface is formulated and its character as an analytic minimal resolution of Kleininan singularities is stated. In §4 the dynamical system on the cubic surface representing the nonlinear monodromy of Painlevé VI is formulated and some preliminary properties of it are given. In §5 we briefly review Bäcklund transformations and their relation to the Riemann-Hilbert correspondence. In §6 fixed points and periodic points of the dynamical system are discussed. A stratification of the parameter space $\mathcal{K}$ is also introduced in order to describe the singularities of the cubic surfaces. In §7 a case-by-case study of fixed points and periodic points is made according to the stratification, thereby a pinpoint identification of finite branch solutions is made on each stratum. In §8 power geometry of algebraic differential equations is applied to Painlevé VI in order to construct as many algebraic branch solutions as possible. In §9 we consider the inclusion of those solutions constructed in §8 into the moduli space of all finite branch solutions. After some preliminaries on Riccati solutions, we show that this inclusion is in fact a surjection, thereby complete the proof of Theorem 1.3.

singularities	$t_1 = 0$	$t_2 = z$	$t_3 = 1$	$t_4 = \infty$
first exponent	$-\lambda_1$	$-\lambda_2$	$-\lambda_3$	$-\lambda_4$
second exponent	λ_1	λ_2	λ_3	$\lambda_4 - 1$
difference	κ_1	κ_2	κ_3	κ_4

Table 1: Riemann scheme: κ_i is the difference of the second exponent from the first.

2 Phase Space

Equation (1) is only a fragmentary appearance of a more intrinsic object constructed algebro-geometrically [16, 17, 18]. We review this construction following the expositions of [20, 21]. The sixth Painlevé dynamical system $\mathrm{P}_{\mathrm{VI}}(\kappa)$ is formulated as a holomorphic, uniform, transversal foliation on a fibration of certain smooth quasi-projective rational surfaces

$$\pi_\kappa : \mathcal{M}(\kappa) \to Z := \mathbb{P}^1 - \{0, 1, \infty\},$$

whose fiber $\mathcal{M}_z(\kappa) := \pi_\kappa^{-1}(z)$ over $z \in Z$, called the *space of initial conditions* at time z, is realized as a moduli space of stable parabolic connections. The total space $\mathcal{M}(\kappa)$ is called the *phase space* of $\mathrm{P}_{\mathrm{VI}}(\kappa)$. In this formulation, the uniformity of the Painlevé foliation, in other words, the geometric Painlevé property of it is a natural consequence of a solution to the Riemann-Hilbert problem (see Theorem 3.5), especially of the properness of the Riemann-Hilbert correspondence [16]. Then equation (1) is just a coordinate expression of the foliation on an affine open subset of $\mathcal{M}(\kappa)$ and the analytic Painlevé property for equation (1) is an immediate consequence of the geometric Painlevé property for the foliation and the algebraicity of the phase space $\mathcal{M}(\kappa)$. Moreover there exists a natural compactification $\mathcal{M}_z(\kappa) \hookrightarrow \overline{\mathcal{M}}_z(\kappa)$ of the moduli space $\mathcal{M}_z(\kappa)$ into a moduli space $\overline{\mathcal{M}}_z(\kappa)$ of stable parabolic phi-connections.

Here we include a very sketchy explanation of the terminology used in the last paragraph. A *stable parabolic connection* is a Fuchsian connection equipped with a parabolic structure on a (rank 2) vector bundle over $\mathbb{P}^1$ having a Riemann scheme as in Table 1, where the parabolic structure corresponds to the first exponents, which satisfies a sort of stability condition in geometric invariant theory. Here the parameter κ_i stands for the difference of the second exponent from the first one at the regular singular point t_i. On the other hand, a *stable parabolic phi-connection* is a variant of stable parabolic connection allowing a "matrix-valued Planck constant" called a phi-operator ϕ such that the generalized Leibniz rule

$$\nabla(fs) = df \otimes \phi(s) + f\nabla(s)$$

is satisfied, where the key point here is that the field ϕ may be degenerate or simi-classical. Then the moduli space $\mathcal{M}_z(\kappa)$ can be compactified by adding some semi-classical objects, that is, some stable parabolic phi-connections with degenerate phi-operator ϕ. There is the following characterization of our moduli spaces (see Figure 3).

Theorem 2.1 ([16, 17, 18])

(1) *The compactified moduli space $\overline{\mathcal{M}}_z(\kappa)$ is isomorphic to an 8-point blow-up of the Hirzebruch surface $\Sigma_2 \to \mathbb{P}^1$ of degree 2.*

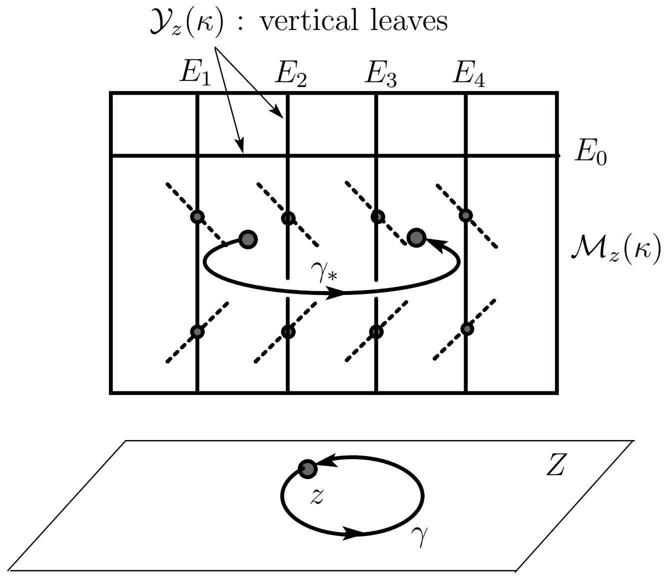


Figure 3: Nonlinear monodromy $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ along a loop $\gamma \in \pi_1(Z, z)$

(2) $\overline{\mathcal{M}}_z(\kappa)$ *has a unique effective anti-canonical divisor* $\mathcal{Y}_z(\kappa)$, *which is given by*

$$\mathcal{Y}_z(\kappa) = 2E_0 + E_1 + E_2 + E_3 + E_4, \tag{5}$$

where E_0 *is the strict transform of the section at infinity and* E_i $(i = 1, 2, 3, 4)$ *is the strict transform of the fiber over the point* $t_i \in \mathbb{P}^1$ *of the Hirzebruch surface* $\Sigma_2 \to \mathbb{P}^1$.

(3) *The support of the divisor* $\mathcal{Y}_z(\kappa)$ *is exactly the locus where the phi-operator* ϕ *is degenerate, with the coefficients of formula* (5) *being the ranks of degeneracy of* ϕ. *In particular,*

$$\mathcal{M}_z(\kappa) = \overline{\mathcal{M}}_z(\kappa) - \mathcal{Y}_z(\kappa).$$

This theorem implies that $\mathcal{M}_z(\kappa)$ is a moduli-theoretical realization of the space of initial conditions for $\mathrm{P_{VI}}(\kappa)$ constructed "by hands" in [26], $\overline{\mathcal{M}}_z(\kappa)$ is a generalized Halphen surface of type $D_4^{(1)}$ in [30] and $(\overline{\mathcal{M}}_z(\kappa), \mathcal{Y}_z(\kappa))$ is an Okamoto-Painlevé pair of type $\widetilde{D}_4$ in [28].

Since the Painlevé foliation has the geometric Painlevé property [16], each loop $\gamma \in \pi_1(Z, z)$ admits global horizontal lifts along the foliation and induces an automorphism

$$\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa), \quad Q \mapsto Q', \tag{6}$$

called the *nonlinear monodromy* along the loop γ (see Figure 3). Note that a fixed point or a periodic point of the map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ can be identified with a solution germ at z which is single-valued or finitely many-valued along the loop γ, respectively.

3 Riemann-Hilbert Correspondence

Generally speaking, a Riemann-Hilbert correspondence is the map from a moduli space of flat connections to a moduli space of monodromy representations, sending a connection to its

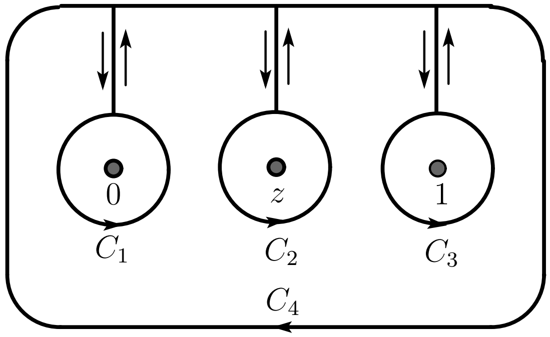


Figure 4: Four loops in $\mathbb{P}^1 - \{0, z, 1, \infty\}$

monodromy. In our situation an appropriate Riemann-Hilbert correspondence

$$\mathrm{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \mathcal{R}_z(a), \qquad Q \mapsto \rho, \tag{7}$$

is formulated in [16, 17, 18]. For each $a = (a_1, a_2, a_3, a_4) \in A := \mathbb{C}^4$, let $\mathcal{R}_z(a)$ denote the moduli space of Jordan equivalence classes of linear monodromy representations

$$\rho : \pi_1(\mathbb{P}^1 - \{0, z, 1, \infty\}, *) \to SL_2(\mathbb{C}),$$

with the prescribed local monodromy data $\mathrm{Tr}\, \rho(C_i) = a_i$ $(i = 1, 2, 3, 4)$, where C_i is a loop as in Figure 4. Any stable parabolic connection $Q \in \mathcal{M}_z(\kappa)$, restricted to $\mathbb{P}^1 - \{0, z, 1, \infty\}$, induces a flat connection and determines the Jordan equivalence class $\rho \in \mathcal{R}_z(a)$ of its monodromy representations, where the correspondence of parameters $\kappa \mapsto a$ is described as follows. If

$$b_i = \begin{cases} \exp(\sqrt{-1}\pi\kappa_i) & (i = 0, 1, 2, 3), \\ -\exp(\sqrt{-1}\pi\kappa_4) & (i = 4), \end{cases} \tag{8}$$

then $b = (b_0, b_1, b_2, b_3, b_4)$ belongs to the multiplicative space

$$B := \{\, b = (b_0, b_1, b_2, b_3, b_4) \in (\mathbb{C}^\times)^5 \;:\; b_0^2 b_1 b_2 b_3 b_4 = 1 \,\}.$$

The Riemann scheme in Table 1 then implies that the monodromy matrix $\rho(C_i)$ has an eigenvalue b_i for each $i = 1, 2, 3, 4$. Since $\rho(C_i) \in SL_2(\mathbb{C})$, its trace $a_i = \mathrm{Tr}\, \rho(C_i)$ is given by

$$a_i = b_i + b_i^{-1} \qquad (i = 1, 2, 3, 4). \tag{9}$$

Given any $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4_\theta$, consider the affine cubic surface

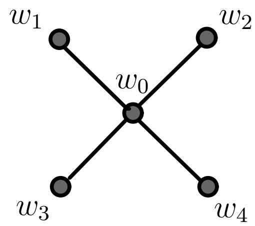
$$\mathcal{S}(\theta) = \{x \in \mathbb{C}^3_x \;:\; f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4 = 0\}.$$

Then there exists an isomorphism of affine algebraic surfaces

$$\mathcal{R}_z(a) \to \mathcal{S}(\theta), \quad \rho \mapsto x = (x_1, x_2, x_3), \quad \text{with} \quad x_i = \mathrm{Tr}\, \rho(C_j C_k)$$

for $\{i, j, k\} = \{1, 2, 3\}$, where the correspondence of parameters $a \mapsto \theta$ is given by

$$\theta_i = \begin{cases} a_i a_4 + a_j a_k & (\{i, j, k\} = \{1, 2, 3\}), \\ a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 & (i = 4). \end{cases} \tag{10}$$



$$C = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Figure 5: Dynkin diagram and Cartan matrix of type $D_4^{(1)}$

The composition of the sequence $\kappa \mapsto b \mapsto a \mapsto \theta$ of the three maps (8), (9) and (10) is referred to as the *Riemann-Hilbert correspondence in the parameter level* [16] and is denoted by

$$\mathrm{rh} : \mathcal{K} \to \Theta. \tag{11}$$

Then the Riemann-Hilbert correspondence (7) is reformulated as a holomorphic map

$$\mathrm{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \mathcal{S}(\theta) \qquad \text{with} \quad \theta = \mathrm{rh}(\kappa). \tag{12}$$

The map (11) admits a remarkable affine Weyl group structure [16, 19], from which the Bäcklund transformations of Painlevé VI emerge [15]. In view of formula (2) the affine space $\mathcal{K}$ can be identified with the linear space $\mathbb{C}^4$ by the forgetful isomorphism $\mathcal{K} \to \mathbb{C}^4$, $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$, where the latter space $\mathbb{C}^4$ is equipped with the standard (complex) Euclidean inner product. For each $i \in \{0, 1, 2, 3, 4\}$, let $w_i : \mathcal{K} \to \mathcal{K}$, $\kappa \mapsto \kappa'$, be the orthogonal reflection in the hyperplane $\{\, \kappa \in \mathcal{K} \,:\, \kappa_i = 0\}$, which is explicitly represented as

$$\kappa'_j = \kappa_j + \kappa_i c_{ij} \qquad (i, j \in \{0, 1, 2, 3, 4\}), \tag{13}$$

where $C = (c_{ij})$ is the Cartan matrix of type $D_4^{(1)}$ given in Figure 5. Then the group generated by w_0, w_1, w_2, w_3, w_4 is an affine Weyl group of type $D_4^{(1)}$,

$$W(D_4^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \curvearrowright \mathcal{K}.$$

corresponding to the Dynkin diagram in Figure 5. The reflecting hyperplanes of all reflections in the group $W(D_4^{(1)})$ are given by affine linear relations

$$\kappa_i = m, \qquad \kappa_1 \pm \kappa_2 \pm \kappa_3 \pm \kappa_4 = 2m + 1 \qquad (i \in \{1, 2, 3, 4\},\ m \in \mathbb{Z}),$$

where the signs $\pm$ may be chosen arbitrarily. Let **Wall** be the union of all these hyperplanes. Then the affine Weyl group structure on (11) is stated as follows [16] (see Figure 6).

Lemma 3.1 *In terms of $b \in B$, the discriminant $\Delta(\theta)$ of the cubic surfaces $\mathcal{S}(\theta)$ factors as*

$$\Delta(\theta) = \prod_{l=1}^{4} (b_l - b_l^{-1})^2 \prod_{\varepsilon \in \{\pm 1\}^4} (b^{\varepsilon} - 1), \tag{14}$$

where we put $b^{\varepsilon} = b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3} b_4^{\varepsilon_4}$ for each quadruple sign $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4$. The Riemann-Hilbert correspondence in the parameter level (11) is a branched $W(D_4^{(1)})$-covering ramifying along **Wall** *and mapping* **Wall** *onto the discriminant locus $\Delta(\theta) = 0$ in Θ.*

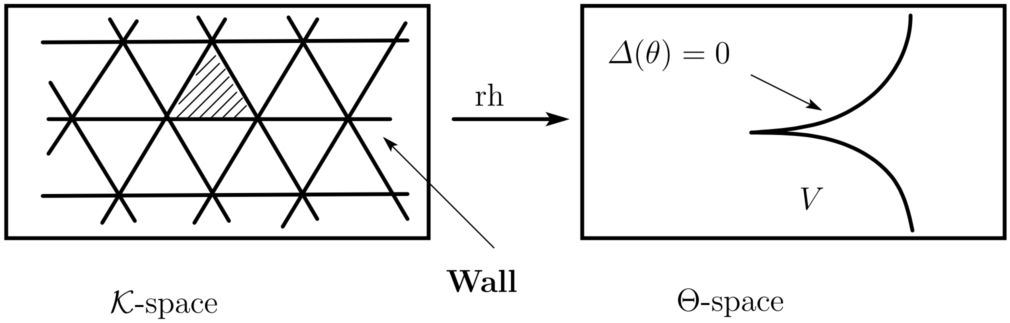


Figure 6: Riemann-Hilbert correspondence in the parameter level

The singularity structure of the cubic surfaces $\mathcal{S}(\theta)$ can be described in terms of the stratification of $\mathcal{K}$ by proper Dynkin subdiagrams, which we now define.

Definition 3.2 Let $\mathcal{I}$ be the set of all *proper* subsets of $\{0,1,2,3,4\}$ including the empty set $\emptyset$. For each element $I \in \mathcal{I}$, we put

$$\begin{aligned} \overline{\mathcal{K}}_I &= \text{the } W(D_4^{(1)})\text{-translates of the set } \{\, \kappa \in \mathcal{K} \,:\, \kappa_i = 0 \ (i \in I)\,\}, \\ D_I &= \text{the Dynkin subdiagram of } D_4^{(1)} \text{ that has nodes } \bullet \text{ exactly in } I. \end{aligned}$$

Let $\mathcal{K}_I$ be the set obtained from $\overline{\mathcal{K}}_I$ by removing the sets $\overline{\mathcal{K}}_J$ with $\#J = \#I + 1$. Then it turns out that we have either $\mathcal{K}_I = \mathcal{K}_{I'}$ or $\mathcal{K}_I \cap \mathcal{K}_{I'} = \emptyset$ for any distinct subsets $I,\, I' \in \mathcal{I}$ (see Remark 3.3). So we can think of the stratification of $\mathcal{K}$ by the subsets $\mathcal{K}_I$ $(I \in \mathcal{I})$, called the $W(D_4^{(1)})$-*stratification*, where each $\mathcal{K}_I$ is referred to as a $W(D_4^{(1)})$-*stratum*. For example, if $I = \emptyset$, one has the *big open* $\mathcal{K}_\emptyset = \mathcal{K} - \textbf{Wall}$. Other examples of $W(D_4^{()1})$-strata are given in Figure 7. The diagram D_I encodes not only its underlying *abstract* Dynkin type but also the inclusion pattern $D_I \hookrightarrow D_4^{(1)}$, a kind of marking. The abstact Dynkin type of D_I is denoted by $\mathrm{Dynk}(I)$. All the feasible abstract Dynkin types are tabulated in Table 2.

There is a mistake in the definition of $\mathcal{K}_I$ in [16, Definition 9.3] and [21], which is now corrected in Definition 3.2. (As for [16], correction may be possible before it is published.)

Remark 3.3 Let I and I' be distinct elements of $\mathcal{I}$. If $\mathrm{Dynk}(I) \neq \mathrm{Dynk}(I')$, then $\mathcal{K}_I \cap \mathcal{K}_{I'} = \emptyset$. On the other hand, if $\mathcal{K}_I = \mathcal{K}_{I'}$ then $\mathrm{Dynk}(I) = \mathrm{Dynk}(I')$ must be of abstract type A_1 or A_2.

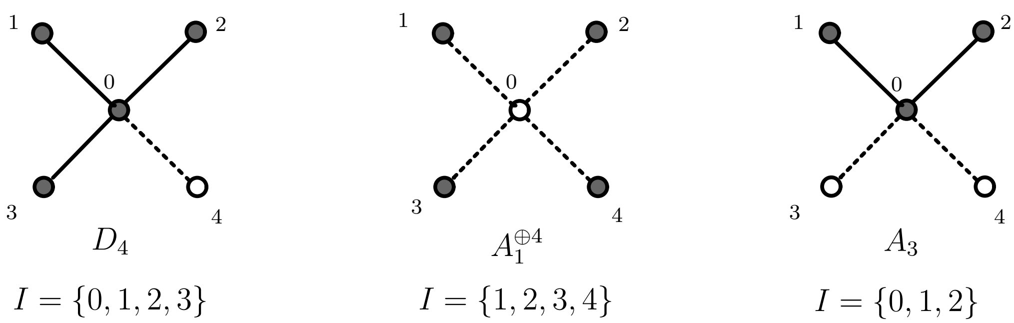


Figure 7: Examples of $W(D_4^{(1)})$-strata

number of nodes	4	3	2	1	0
abstract	D_4	A_3	A_2	A_1	$\emptyset$
Dynkin type	$A_1^{\oplus 4}$	$A_1^{\oplus 3}$	$A_1^{\oplus 2}$	$-$	$-$

Table 2: Feasible abstract Dynkin types

(1) There is a unique $W(D_4^{(1)})$-stratum of abstract type $\emptyset$, or A_1, or A_2, or $A_1^{\oplus 4}$.

(2) There are six $W(D_4^{(1)})$-strata of abstract type $A_1^{\oplus 2}$ or A_3.

(3) There are four $W(D_4^{(1)})$-strata of abstract type $A_1^{\oplus 3}$ or D_4.

Example 3.4 We consider the $W(D_4^{(1)})$-strata of abstract types $A_1^{\oplus 4}$ and D_4.

(1) The unique $W(D_4^{(1)})$-stratum of abstract type $A_1^{\oplus 4}$ exactly corresponds to the value $\theta = (0, 0, 0, -4)$. A parameter $\kappa \in \mathcal{K}$ lies in this stratum if and only if either

  (a) $\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4 \in \mathbb{Z}$, $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \in 2\mathbb{Z}$; or

  (b) $\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4 \in \mathbb{Z} + 1/2$.

(2) The four $W(D_4^{(1)})$-strata of abstract type D_4 exactly correspond to the values $\theta = (8\varepsilon_1, 8\varepsilon_2, 8\varepsilon_3, 28)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm\}^3$ ranges over all triple signs such that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. A parameter $\kappa \in \mathcal{K}$ lies in the union of these $W(D_4^{(1)})$-strata if and only if

$$\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{Z}, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \in 2\mathbb{Z} + 1.$$

With this stratification, we have a very neat solution to the Riemann-Hilbert problem.

Theorem 3.5 ([16, 17, 18]) *Given any $\kappa \in \mathcal{K}$, put $\theta = \mathrm{rh}(\kappa) \in \Theta$. Then,*

(1) *if $\kappa \in \mathcal{K}_I$ then $\mathcal{S}(\theta)$ has Kleinian singularities of Dynkin type D_I,*

(2) *the Riemann-Hilbert correspondence (12) is a proper surjective map that is an analytic minimal resolution of Kleinian singularities.*

If $\kappa \in \mathcal{K} - \mathbf{Wall}$ then the surface $\mathcal{S}(\theta)$ is smooth and $\mathrm{RH}_{z,\kappa}$ is a biholomorphism, while if $\kappa \in \mathbf{Wall}$, it is not a biholomorphism but only gives a resolution of singularities (proper and surjective, but not injective). For example, see Figure 8 for the case $\kappa = (0, 0, 0, 0, 1)$ where a singularity of type D_4 occurs. In the latter case, however, if we take a standard *algebraic* minimal resolution of Kleinian singularities as constructed by Brieskorn [4] and others,

$$\varphi : \widetilde{\mathcal{S}}(\theta) \to \mathcal{S}(\theta) \tag{15}$$

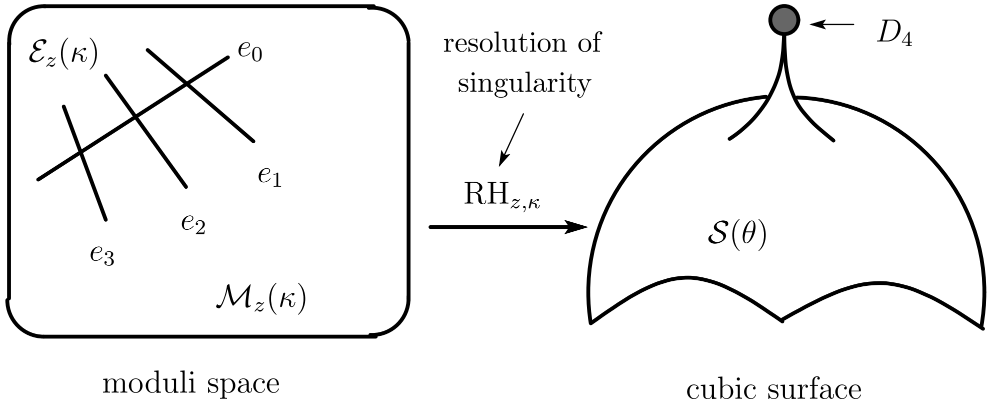


Figure 8: Resolution of singularities by Riemann-Hilbert correspondence

then we can lift the Riemann-Hilbert correspondence (12) to have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_z(\kappa) & \xrightarrow{\widetilde{\mathrm{RH}}_{z,\kappa}} & \widetilde{\mathcal{S}}(\theta) \\ \| & & \downarrow \varphi \\ \mathcal{M}_z(\kappa) & \xrightarrow{\mathrm{RH}_{z,\kappa}} & \mathcal{S}(\theta). \end{array} \tag{16}$$

The lifted Riemann-Hilbert correspondence $\widetilde{\mathrm{RH}}_{z,\kappa}$ is a biholomorphism and hence gives a strict conjugacy between the nonlinear monodromy (6) of $\mathrm{P_{VI}}(\kappa)$ and a certain automorphism

$$\tilde{g} : \widetilde{\mathcal{S}}(\theta) \to \widetilde{\mathcal{S}}(\theta). \tag{17}$$

This latter map will be described explicitly in Section 4 (see Theorem 4.1).

The singularity structure of the affine cubic surface $\mathcal{S}(\theta)$ is closely related to the Riccati solutions to $\mathrm{P_{VI}}(\kappa)$ [16], where a Riccati solution is a particular solution that arises from the Riccati equation associated to a Gauss hypergeometric equation. Let $\mathcal{E}_z(\kappa) \subset \mathcal{M}_z(\kappa)$ be the exceptional set of the resolution of singularities by the Riemann-Hilbert correspondence (12). Similarly, let $\mathcal{E}(\theta) \subset \widetilde{\mathcal{S}}(\theta)$ be the exceptional set of the algebraic resolution of singularities (15).

Theorem 3.6 ([16, 29]) *Equation* $\mathrm{P_{VI}}(\kappa)$ *admits Riccati solutions if and only if* $\kappa \in \mathbf{Wall}$. *All Riccati solution germs at time* $z \in Z$ *are parametrized by the exceptional set* $\mathcal{E}_z(\kappa) \subset \mathcal{M}_z(\kappa)$, *which precisely corresponds to the exceptional set* $\mathcal{E}(\theta) \subset \widetilde{\mathcal{S}}(\theta)$ *through the lifted Riemann-Hilbert correspondence* (*16*).

Fot this reason we may refer to $\mathcal{E}_z(\kappa)$ and $\mathcal{M}_z^\circ(\kappa) := \mathcal{M}_z(\kappa) - \mathcal{E}_z(\kappa)$ as the *Riccati locus* and *non-Riccati locus* of $\mathcal{M}_z(\kappa)$ respectively. They are invariant under the action of the nonlinear monodromy (6). Corresponding to them, let $\mathrm{Sing}(\theta)$ and $\mathcal{S}^\circ(\theta) := \mathcal{S}(\theta) - \mathrm{Sing}(\theta)$ be the singular locus and the smooth locus of the cubic surface $\mathcal{S}(\theta)$ respectively.

Remark 3.7 Two remarks are in order at this stage.

(1) By Theorem 3.5 the Riemann-Hilbert correspondence (12) restricts to a biholomorphism

$$\mathrm{RH}_{z,\kappa}^\circ : \mathcal{M}_z^\circ(\kappa) \to \mathcal{S}^\circ(\theta) \tag{18}$$

between the non-Riccati locus of $\mathcal{M}_z(\kappa)$ and the smooth locus of $\mathcal{S}(\theta)$, while it collapses the Riccati locus $\mathcal{E}_z(\kappa)$ to the singular locus $\mathrm{Sing}(\theta)$. In order to resolve this degeneracy and obtain an isomorphism, we had to take the lifted Riemann-Hibert correspondence (16), which induces an isomorphism between the exceptional sets $\mathcal{E}_z(\kappa)$ and $\mathcal{E}(\theta)$.

(2) For the Riccati solutions the main problem of this article is trivial; if a Riccati solution is a finite branch solution around a fixed singular point, then it is an algebraic branch solution, because the Riccati solution is (essentially) the logarithmic derivative of a Gauss hypergeometric function. Thus we may restrict our attention to the non-Riccati locus.

4 Dynamics on Cubic Surface

We shall describe the strict conjugacy (17) of the nonlinear monodromy (6). For a cyclic permutation (i, j, k) of $(1, 2, 3)$ we define an isomorphism $g_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta')$, $(x, \theta) \mapsto (x', \theta')$ by

$$g_i : (x'_i, x'_j, x'_k, \theta'_i, \theta'_j, \theta'_k) = (\theta_j - x_j - x_k x_i, x_i, x_k, \theta_j, \theta_i, \theta_k). \tag{19}$$

Through the resolution of singularities (15), the map g_i is uniquely lifted to an isomorphism

$$\tilde{g}_i : \widetilde{\mathcal{S}}(\theta) \to \widetilde{\mathcal{S}}(\theta'), \qquad (i = 1, 2, 3).$$

We remark that the square g_i^2 is an automorphism of $\mathcal{S}(\theta)$ with $\tilde{g}_i^2$ being its lift to $\widetilde{\mathcal{S}}(\theta)$.

Theorem 4.1 ([16]) *For each $i \in \{1, 2, 3\}$ the nonlinear monodromy $\gamma_{i*} : \mathcal{M}_z(\kappa) \circlearrowleft$ along the i-th basic loop γ_i is strictly conjugated to the automorphism $\tilde{g}_i^2 : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ via the lifted Riemann-Hilbert correspondence (16). More generally, if $\gamma \in \pi_1(Z, z)$ is represented by $\gamma = \gamma_{i_1}^{\varepsilon_1}\gamma_{i_2}^{\varepsilon_2}\cdots\gamma_{i_n}^{\varepsilon_n}$ with $(i_1, \dots, i_n) \in \{1, 2, 3\}^n$ and $(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$, then the map (17) is given by*

$$\tilde{g} = \tilde{g}_{i_1}^{2\varepsilon_1}\tilde{g}_{i_2}^{2\varepsilon_2}\cdots\tilde{g}_{i_n}^{2\varepsilon_n}.$$

Let $\widetilde{\mathrm{Fix}}_j(\theta)$ be the set of all fixed points of the transformation $\tilde{g}_j^2 : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$. Moreover, for any integer $n > 1$, let $\widetilde{\mathrm{Per}}_j(\theta; n)$ be the set of all periodic points of *prime* period n of the transformation $\tilde{g}_j^2 : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$. Theorem 4.1 then implies that all single-valued solution germs and all n-branch solution germs to $\mathrm{P_{VI}}(\kappa)$ around the fixed singular point z_j are parametrized by the sets $\widetilde{\mathrm{Fix}}_j(\theta)$ and $\widetilde{\mathrm{Per}}_j(\theta; n)$ respectively. By Remark 3.7, considering $\widetilde{\mathrm{Fix}}_j(\theta)$ and $\widetilde{\mathrm{Per}}_j(\theta; n)$ upstairs is the same thing as considering $\mathrm{Fix}_j(\theta)$ and $\mathrm{Per}_j(\theta; n)$ downstairs, except for the exceptional locus upstairs and the singular locus downstairs. Here $\mathrm{Fix}_j(\theta)$ and $\mathrm{Per}_j(\theta; n)$ denote the set of all fixed points and the set of all periodic points of prime period n of the transformation $g_j^2 : \mathcal{S}(\theta) \circlearrowleft$ downstairs. In order to make the situation more transparent, we begin by investigating simultaneous fixed points of g_1^2, g_2^2, g_3^2 downstairs.

Theorem 4.2 *If $\mathrm{Fix}(\theta)$ is the set of all simultaneous fixed points of g_1^2, g_2^2, $g_3^2 : \mathcal{S}(\theta) \circlearrowleft$, then*

$$\mathrm{Fix}(\theta) = \mathrm{Sing}(\theta). \tag{20}$$

Proof. A point $x \in \mathcal{S}(\theta)$ is a singular point of the surface $\mathcal{S}(\theta)$ if and only if its gradient vector field $y(x,\theta) = (y_1(x,\theta), y_2(x,\theta), y_3(x,\theta))$ vanishes at the point x, where

$$y_i(x,\theta) := \frac{\partial f}{\partial x_i}(x,\theta) = 2x_i + x_j x_k - \theta_i \tag{21}$$

On the other hand, an inspection of formula (19) readily shows that $x \in \mathcal{S}(\theta)$ is a simultaneous fixed point of g_1^2, g_2^2, g_3^2 if and only if x is a common root of equations

$$f(x,\theta) = y_1(x,\theta) = y_2(x,\theta) = y_3(x,\theta) = 0. \tag{22}$$

Then the equality (20) immediately follows from these observations. □

As is announced in [16], this theorem yields a characterization of the rational solutions.

Corollary 4.3 *Any single-valued global solution to* $\mathrm{P_{VI}}(\kappa)$ *is a rational Riccati solution.*

Proof. If a single-valued solution $Q \in \mathcal{M}_z(\kappa)$ belongs to the non-Riccati locus $Q \in \mathcal{M}_z^\circ(\kappa)$, then the Riemann-Hilbert correspondence (18) sends Q to a smooth point $x \in \mathcal{S}^\circ(\theta)$. Since the single-valued solution Q is a simultaneous fixed point of the nonlinear monodromies γ_{1*}, γ_{2*}, γ_{3*}, the corresponding point x must lie in $\mathrm{Fix}(\theta)$. Then Theorem 4.2 implies that $x \in \mathrm{Sing}(\theta)$, which contradicts the fact that $x \in \mathcal{S}^\circ(\theta)$. Hence any single-valued solution is a Riccati solution. Since any Riccati solution is (essentially) the logarithmic derivative of a Gauss hypergeometric function, any single-valued Riccati solution must be a rational solution. □

All the rational solutions to Painlevé VI are classified in [25]. We come back to our discussion downstairs and give a simple characterization of the sets $\mathrm{Fix}_j(\theta)$ and $\mathrm{Per}_j(\theta; n)$.

Lemma 4.4 *Let* $x = (x_1, x_2, x_3) \in \mathcal{S}(\theta)$ *be any point and let* n *be any integer* > 1.

(1) $x \in \mathrm{Fix}_j(\theta)$ *if and only if* x *is a root of equations*

$$f(x,\theta) = y_j(x,\theta) = y_k(x,\theta) = 0. \tag{23}$$

(2) $x \in \mathrm{Per}_j(\theta; n)$ *if and only if there exists an integer* $0 < m < n$ *coprime to* n *such that*

$$f(x,\theta) = 0, \qquad x_i = 2\cos(\pi m/n). \tag{24}$$

Proof. We put $(x', \theta') = g_j(x,\theta)$ and $y' = y(x', \theta')$. Then formula (19) yields

$$y_i' = y_i - x_j y_k, \qquad y_j' = -y_k, \qquad y_k' = y_j - x_i y_k. \tag{25}$$

For each integer $n \in \mathbb{Z}$ we write $(x^{(n)}, \theta^{(n)}) = g_i^n(x,\theta)$ and $y^{(n)} = y(x^{(n)}, \theta^{(n)})$. From formulas (19) and (25), we can easily obtain three recurrence relations

$$y_j^{(n+2)} + x_i\, y_j^{(n+1)} + y_j^{(n)} = 0, \tag{26}$$

$$x_j^{(n+2)} - x_j^{(n)} = y_j^{(n+2)}, \tag{27}$$

$$x_k^{(n+1)} = x_j^{(n)}. \tag{28}$$

The characteristic equation of the recurrence relation (26) is the quadratic equation

$$\lambda^2 + x_i \lambda + 1 = 0, \tag{29}$$

the roots of which are denoted by α and $\beta = \alpha^{-1}$. Since $\alpha\beta = 1$, we may and shall assume that $|\alpha| \ge 1 \ge |\beta| > 0$ in the sequel. The discussion is divided into two cases.

Case $\boldsymbol{x_i} \in \mathbb{C} - \{\pm 2\}$: In this case, the roots α and β are distinct and different from ± 1 and the recurrence relation (26) is settled as

$$y_j^{(n)} = \frac{\beta^n(\alpha y_j + y_k) - \alpha^n(\beta y_j + y_k)}{\alpha - \beta}.$$

Then it follows from (27) and (28) that the sequences $x_j^{(n)}$ and $x_k^{(n)}$ are determined as

$$\begin{cases} x_j^{(2n)} = x_k^{(2n+1)} = p\,\alpha^{2n} + q\,\beta^{2n} + r_1, \\ x_j^{(2n+1)} = x_k^{(2n)} = p\,\alpha^{2n+1} + q\,\beta^{2n+1} + r_2, \end{cases} \tag{30}$$

where the constants p, q, r_1 and r_2 are given by

$$p = -\frac{\alpha^2(\beta y_j + y_k)}{(\alpha-\beta)(\alpha^2-1)}, \qquad q = \frac{\beta^2(\alpha y_j + y_k)}{(\alpha-\beta)(\beta^2-1)},$$
$$r_1 = x_j - p - q, \qquad r_2 = x_j' - \alpha p - \beta q.$$

Notice that $p = q = 0$ if and only if x satisfies equations (23). Indeed, the condition $p = q = 0$ is equivalent to $\alpha y_j + y_k = \beta y_j + y_k = 0$, which is equivalent to the condition $y_j = y_k = 0$, because the roots α and β are distinct.

Now we assume that x is a root of equations (23). Then (30) implies that the sequence $x^{(n)}$ is periodic of period two, that is, x is a fixed point of g_j^2. Next we assume that x is not a root of equations (23). If x is a periodic point of g_j^2 of prime period $n \ge 1$, then (30) yields

$$x_j^{(2n)} - x_j = (\alpha^{2n} - 1)(p - q\beta^{2n}) = 0,$$
$$x_k^{(2n)} - x_k = (\alpha^{2n} - 1)(p\alpha - q\beta^{2n+1}) = 0.$$

Here it cannot happen that $p - q\beta^{2n} = p\alpha - q\beta^{2n+1} = 0$. Indeed, otherwise, we have $p = q\beta^{2n}$ and $q(1-\beta^2) = 0$. Since at least one of p and q is nonzero, we have $\beta \in \{\pm 1\}$ and hence $x_i \in \{\pm 2\}$, which contradicts the assumption that $x_i \not\in \{\pm 2\}$. Therefore, $\alpha^{2n} = 1$, that is, α is a primitive $2n$-th root of unity. Note that $n \ge 2$ since $\alpha \not\in \{\pm 1\}$. Thus there is an integer $0 < m < n$ comprime to n such that $\alpha = \exp(\pi i m/n)$ and so $x_i = \alpha + \alpha^{-1} = 2\cos(\pi m/n)$, which leads to condition (24). Conversely, if condition (24) is satisfied, then it is easy to see that x is a periodic point of g_j^2 of prime period n.

Case $\boldsymbol{x_i} \in \{\pm 2\}$: In this case we have $x_i = -2\varepsilon$ for some sign $\varepsilon \in \{\pm 1\}$ and hence equation (29) has a double root $\alpha = \beta = \varepsilon$. Then the recurrence equation (26) is settled as $y_j^{(n)} = \varepsilon^n\{y_j - n(y_j + \varepsilon y_k)\}$. If the sequence $x^{(n)}$ is periodic, then so is the sequence $y_j^{(n)}$. This is the case if and only if $y_j + \varepsilon y_k = 0$. Conversely, if this condition is satisfied, then we have $y_j^{(n)} = \varepsilon^n y_j$. Substituting this equation into (27) yields

$$\begin{cases} x_j^{(2n)} = x_k^{(2n+1)} = x_j + n y_j, \\ x_j^{(2n+1)} = x_k^{(2n+2)} = x_j' + \varepsilon n y_j. \end{cases} \tag{31}$$

Hence the sequence $x^{(2n)}$ is periodic if and only if $y_j = y_k = 0$, namely, if and only if x is a root of (23). In this case x is a fixed point of g_j^2. $\square$

In order to give the relation between the fixed points upstairs and those downstairs, we put

$$\mathrm{Fix}_j^{\circ}(\theta) := \mathrm{Fix}_j(\theta) - \mathrm{Sing}(\theta), \quad \widetilde{\mathrm{Fix}}{}_j^{\circ}(\theta) := \widetilde{\mathrm{Fix}}_j(\theta) - \mathcal{E}(\theta), \quad \widetilde{\mathrm{Fix}}{}_j^{e}(\theta) := \widetilde{\mathrm{Fix}}_j(\theta) \cap \mathcal{E}(\theta).$$

For the periodic points of prime period $n > 1$, we define $\mathrm{Per}_j^{\circ}(\theta; n)$, $\widetilde{\mathrm{Per}}{}_j^{\circ}(\theta; n)$ and $\widetilde{\mathrm{Per}}{}_j^{e}(\theta; n)$ in a similar manner. Then there exist direct sum decompositions

$$\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\mathrm{Fix}}{}_j^{\circ}(\theta) \amalg \widetilde{\mathrm{Fix}}{}_j^{e}(\theta), \qquad \widetilde{\mathrm{Per}}_j(\theta; n) = \widetilde{\mathrm{Per}}{}_j^{\circ}(\theta; n) \amalg \widetilde{\mathrm{Per}}{}_j^{e}(\theta; n),$$

where the exceptional components $\widetilde{\mathrm{Fix}}{}_j^{e}(\theta)$ and $\widetilde{\mathrm{Per}}{}_j^{e}(\theta; n)$ parametrize the single-valued Riccati solutions and the n-branched Riccati solutions around the fixed singular point z_j respectively.

Lemma 4.5 *The minimal resolution* (*15*) *induces an isomorphism*

$$\varphi : \widetilde{\mathrm{Fix}}{}_j^{\circ}(\theta) \to \mathrm{Fix}_j^{\circ}(\theta). \tag{32}$$

For any $n > 1$ we have $\mathrm{Per}(\theta; n) \cap \mathrm{Sing}(\theta) = \emptyset$*, that is,* $\mathrm{Per}_j^{\circ}(\theta; n) = \mathrm{Per}_j(\theta; n)$*, and the minimal resolution* (*15*) *induces an isomorphism*

$$\varphi : \widetilde{\mathrm{Per}}{}_j^{\circ}(\theta; n) \to \mathrm{Per}_j(\theta; n) \qquad (n > 1). \tag{33}$$

Proof. The isomorphism (32) is trivial from the definition. The assertion $\mathrm{Per}(\theta; n) \cap \mathrm{Sing}(\theta) = \emptyset$ follows from (20). Then the isomorphism (33) is again trivial from the definition. $\square$

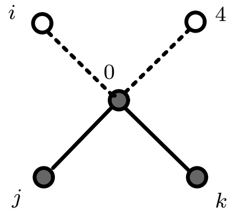
The fixed point set and the periodic point set, upstairs or downstairs, will be investigated more closely in §6. For this purpose it is convenient to consider the symmetric group S_4 of degree 4 acting on $\mathcal{K}$ by permuting the entries κ_1, κ_2, κ_3, κ_4 of $\kappa \in \mathcal{K}$ and fixing κ_0. Through the Riemann-Hilbert correspondence in the parameter level, $\mathrm{rh} : \mathcal{K} \to \Theta$, the action $S_4 \curvearrowright \mathcal{K}$ induces an action of $S_3 \ltimes \mathrm{Kl}$ on Θ, where Kl is Klein's 4-group realized as the group of even triple signs, $\mathrm{Kl} = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3 \,:\, \varepsilon_1\varepsilon_2\varepsilon_3 = 1\}$, acting on Θ by the sign changes $(\theta_1, \theta_2, \theta_3, \theta_4) \mapsto (\varepsilon_1\theta_1, \varepsilon_2\theta_2, \varepsilon_3\theta_3, \theta_4)$, while S_3 acts on Θ by permuting the entries θ_1, θ_2, θ_3 of $\theta \in \Theta$ and fixing θ_4. This construction defines an isomorphism of groups

$$S_4 \cong S_3 \ltimes \mathrm{Kl}, \quad \sigma \mapsto (\tau, \varepsilon), \tag{34}$$

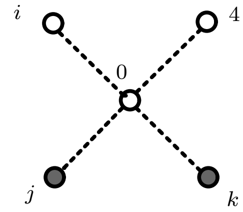
with respect to which the map $\mathrm{rh} : \mathcal{K} \to \Theta$ becomes S_4-equivariant. Viewed as a subgroup of S_4, Klein's 4-group is the permutation group $\mathrm{Kl} = \{1, (14)(23), (24)(31), (34)(12)\}$.

Let $\sigma \in S_4$ act on $x = (x_1, x_2, x_3)$ in the same manner as it does on $(\theta_1, \theta_2, \theta_3)$. Then the polynomial $f(x, \theta)$ is σ-invariant and hence σ induces an isomorphism of algebraic surfaces, $\sigma : \mathcal{S}(\theta) \to \mathcal{S}(\sigma(\theta))$. As for the action $g_j^2 : \mathcal{S}(\theta) \circlearrowleft$, we have the commutative diagram

$$\begin{tikzcd} \mathcal{S}(\theta) \arrow[r, "g_j^2"] \arrow[d, "\sigma"'] & \mathcal{S}(\theta) \arrow[d, "\sigma"] \\ \mathcal{S}(\sigma(\theta)) \arrow[r, "g_{\tau(j)}^2"'] & \mathcal{S}(\sigma(\theta)), \end{tikzcd} \tag{35}$$



$(A_3)_i$



$(A_1^{\oplus 2})_i$

Figure 9: $\widetilde{W}(D_4^{(1)})$-strata $(A_3)_i$ and $(A_1^{\oplus 2})_i$

for any element $\sigma \in S_4$ with $\tau \in S_3$ determined by (34). It induces isomorphisms

$$\sigma : \mathrm{Fix}_j(\theta) \to \mathrm{Fix}_{\tau(j)}(\sigma(\theta)), \qquad \sigma : \mathrm{Per}_j(\theta; n) \to \mathrm{Per}_{\tau(j)}(\sigma(\theta); n),$$

which, via the minimal resolution (15), lift up to isomorphisms

$$\widetilde{\sigma} : \widetilde{\mathrm{Fix}}_j(\theta) \to \widetilde{\mathrm{Fix}}_{\tau(j)}(\sigma(\theta)), \qquad \widetilde{\sigma} : \widetilde{\mathrm{Per}}_j(\theta; n) \to \widetilde{\mathrm{Per}}_{\tau(j)}(\sigma(\theta); n).$$

The action of the symmetric group S_4 on $\mathcal{K}$ mentioned above is just induced from its action on the index set $\{0, 1, 2, 3, 4\}$ fixing the element 0, namely, from the realization of S_4 as the automorphism group of the Dynkin diagram $D_4^{(1)}$. By taking the semi-direct product by the symmetric group S_4 or by Klein's 4-group Kl, we can enlarge the affine Weyl group $W(D_4^{(1)})$ to the affine Weyl group of type $F_4^{(1)}$ or to the extended affine Weyl group of type $D_4^{(1)}$,

$$W(F_4^{(1)}) = S_4 \ltimes W(D_4^{(1)}) \supset \widetilde{W}(D_4^{(1)}) = \mathrm{Kl} \ltimes W(D_4^{(1)}).$$

Definition 4.6 Replacing the group $W(D_4^{(1)})$ with $W(F_4^{(1)})$ in Definition 3.2, we can define a coarser stratification of $\mathcal{K}$ than the $W(D_4^{(1)})$-stratification, called the $W(F_4^{(1)})$-*stratification*. Moreover, replacing $W(D_4^{(1)})$ with $\widetilde{W}(D_4^1)$, we can also think of a stratification of $\mathcal{K}$ intermediate between these two stratifications, called the $\widetilde{W}(D_4^{(1)})$-*stratification*.

The following is the classification of the $W(F_4^{(1)})$-strata and $\widetilde{W}(D_4^{(1)})$-strata.

Lemma 4.7 *For each abstract Dynkin type $*$ in Table 2, there is a unique $W(F_4^{(1)})$-stratum of type $*$. As for the $\widetilde{W}(D_4^{(1)})$-strata, we have the following classification (see also Figure 9).*

(1) *For $* \in \{D_4, A_1^{\oplus 4}, A_1^{\oplus 3}, A_2, A_1, \emptyset\}$, there is a unique $\widetilde{W}(D_4^{(1)})$-stratum of abstract type $*$ and this unique stratum is denoted by the same symbol $*$.*

(2) *For $* \in \{A_3, A_1^{\oplus 2}\}$, there are exactly three $\widetilde{W}(D_4^{(1)})$-strata of abstract type $*$;*

   (a) *for $* = A_3$, the stratum $(A_3)_i$ represented by $I = \{0, j, k\}$ with $\{i, j, k\} = \{1, 2, 3\}$;*

   (b) *for $* = A_1^{\oplus 2}$, the stratum $(A_1^{\oplus 2})_i$ represented by $I = \{j, k\}$ with $\{i, j, k\} = \{1, 2, 3\}$.*

$$
\begin{tikzcd}
 & & (A_1^{\oplus 2})_i \arrow[r] \arrow[ddr] & A_1^{\oplus 3} \arrow[r] \arrow[ddr] & A_1^{\oplus 4} \\
\emptyset \arrow[r] & A_1 \arrow[ur] \arrow[dr] & & & \\
 & & A_2 \arrow[r] & (A_3)_i \arrow[r] & D_4
\end{tikzcd}
$$

Figure 10: Adjacency relations among $\widetilde{W}(D_4^{(1)})$-strata ($i = 1, 2, 3$)

If something about the transformation g_j^2 is discussed for a *fixed* index j, the relevant stratification is the $\widetilde{W}(D_4^{(1)})$-stratification. Namely we may discuss the issue on each $\widetilde{W}(D_4^{(1)})$-stratum, choosing any representative of each $\widetilde{W}(D_4^{(1)})$-orbit, since in the commutative diagram (35) we have $\tau(j) = j$ and hence $g_{\tau(j)}^2 = g_j^2$ for every $\sigma \in \mathrm{Kl}$ (see also Remark 5.1). For two $\widetilde{W}(D_4^{(1)})$-strata, say $*$ and $**$, we write $* \to **$ if the stratum $**$ lies on the boundary of the stratum $*$. All the possible adjacency relations $* \to **$ are depicted in Figure 10. Note that there are no adjacency relations between $(A_1^{\oplus 2})_i$ and $(A_3)_j$ for any distinct i, $j \in \{1, 2, 3\}$.

5 Bäcklund Transformations

In this section we briefly discuss Bäcklund transformations, especially the characterization of them in terms of Riemann-Hilbert correspondence [15, 16]. This topic is included here in order to confirm that our problem may be treated modulo Bäcklund transformations.

For each $\sigma \in S_4$ we define the isomorphism of affine cubic surfaces

$$
\sigma : \mathcal{S}(\theta) \to \mathcal{S}(\sigma(\theta)), \qquad (x_1, x_2, x_3) \mapsto (\varepsilon_{\tau(1)} x_{\tau(1)},\, \varepsilon_{\tau(2)} x_{\tau(2)},\, \varepsilon_{\tau(3)} x_{\tau(3)}),
$$

where $\sigma \in S_4$ is identified with $(\tau, \varepsilon) \in S_3 \ltimes \mathrm{Kl}$ via the isomorphism (34). Consider the natural homomorphism $W(F_4^{(1)}) = S_4 \ltimes W(D_4^{(1)}) \to S_4$, $w \mapsto \sigma$. Since the Riemann-Hilbert correspondence (12) is an analytic minimal resolution of singularities, for each $w \in W(F_4^{(1)})$, there exists an analytic isomorphism $w : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(w(\kappa))$ such that the diagram

$$
\begin{tikzcd}
\mathcal{M}_z(\kappa) \arrow[r, "w"] \arrow[d, "\mathrm{RH}_{z,\kappa}"'] & \mathcal{M}_z(w(\kappa)) \arrow[d, "\mathrm{RH}_{z,w(\kappa)}"] \\
\mathcal{S}(\theta) \arrow[r, "\sigma"'] & \mathcal{S}(\sigma(\theta))
\end{tikzcd}
\tag{36}
$$

is commutative, for any fixed $\kappa \in \mathcal{K}$ with $\theta = \mathrm{rh}(\kappa) \in \Theta$.

The commutative diagram (36) characterizes the Bäcklund transformations of Painlevé VI. Namely the map $w : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(w(\kappa))$ turns out to be algebraic and there are suitable affine coordinates on $\mathcal{M}_z(\kappa)$ and $\mathcal{M}_z(w(\kappa))$ in terms of which the map w can be represented by the usual formula for Bäcklund transformations known as birational canonical transforamtions [27] (see [15, 16] for the precise statement). In other words the Riemann-Hilbert correspondence is equivariant under the Bäcklund transformations and so is our main problem.

Remark 5.1 The S_4-factor of $W(F_4^{(1)}) = S_4 \ltimes W(D_4^{(1)})$ or more strictly the S_3-factor of $S_4 = S_3 \ltimes \mathrm{Kl}$ permutes the three fixed singular points 0, 1 and ∞, while they are fixed by $\widetilde{W}(D_4^{(1)}) =$

$\mathrm{Kl} \ltimes W(D_4^{(1)})$. Hence we may consider our problem only around the origin $z = 0$ and, upon restricting our attention to $z = 0$, we may discuss it modulo the Bäcklund action of $\widetilde{W}(D_4^{(1)})$.

6 Fixed Points and Periodic Points

We shall more closely investigate the fixed point set $\mathrm{Fix}_j(\theta)$, or rather its subset $\mathrm{Fix}_j^\circ(\theta) \cong \widetilde{\mathrm{Fix}}_j^\circ(\theta)$ of smooth fixed points, by solving the system of equations (23). In view of (21) the last two equations in (23) are expressed as a linear system for the unknowns (x_j, x_k),

$$\begin{cases} 2x_j + x_i x_k = \theta_j, \\ x_i x_j + 2x_k = \theta_k, \end{cases} \tag{37}$$

If its determinant $4 - x_i^2$ is nonzero, then system (37) is uniquely settled as

$$x_j = \frac{2\theta_j - x_i\theta_k}{4 - x_i^2}, \qquad x_k = \frac{2\theta_k - x_i\theta_j}{4 - x_i^2}. \tag{38}$$

Substituting (38) into equation $f(x, \theta) = 0$ yields a quartic equation for the unknown x_i,

$$x_i^4 - \theta_i x_i^3 + (\theta_4 - 4)x_i^2 + (4\theta_i - \theta_j\theta_k)x_i + \theta_j^2 + \theta_k^2 - 4\theta_4 = 0. \tag{39}$$

Conversely, if x_i is a root of equation (39) with nonzero $x_i^2 - 4$, then subsituting this into formula (38) yields a root of system (23). The four roots of quartic equation (39) are given by

$$F(b_i, b_4; b_j, b_k), \quad F(b_i, b_4^{-1}; b_j, b_k), \quad F(b_j, b_k; b_i, b_4), \quad F(b_j, b_k^{-1}; b_i, b_4),$$

counted with multiplicities, where $F(b_i, b_4; b_j, b_k)$ is defined by

$$F(b_i, b_4; b_j, b_k) = b_i b_4 + b_i^{-1} b_4^{-1}. \tag{40}$$

We pick up the root $x_i = F(b_i, b_4; b_j, b_k)$. Note that $F(b_i, b_4; b_j, b_k)^2 - 4$ is nonzero precisely when $b_i^2 b_4^2 \neq 1$. If this is the case, then substituting $x_i = F(b_i, b_4; b_j, b_k)$ into formula (38) yields $x_j = G(b_i, b_4; b_j, b_k)$ and $x_k = G(b_i, b_4; b_k, b_j)$, where $G(b_i, b_4; b_j, b_k)$ is defined by

$$G(b_i, b_4; b_j, b_k) = \frac{(b_i + b_4)(b_j + b_k)(b_j b_k + 1)}{2(b_i b_4 + 1) b_j b_k} + \frac{(b_i - b_4)(b_j - b_k)(b_j b_k - 1)}{2(b_i b_4 - 1) b_j b_k}. \tag{41}$$

Therefore, if $P(b_i, b_4; b_j, b_k)$ denotes the point defined by

$$x_i = F(b_i, b_4; b_j, b_k), \qquad x_j = G(b_i, b_4; b_j, b_k), \qquad x_k = G(b_i, b_4; b_k, b_j),$$

then $x = P(b_i, b_4; b_j, b_k)$ gives a root of system (23) with nonzero $x_i^2 - 4$ provided that $b_i^2 b_4^2 \neq 1$. If x is at this root, then $y_i(x, \theta)$ admits the following nice factorization

$$\begin{aligned} y_i(x, \theta) &= \frac{(b_i - b_i^{-1})(b_4 - b_4^{-1})}{(b_i^2 b_4^2 - 1)^2} \prod_{(\varepsilon_j, \varepsilon_k) \in \{\pm 1\}^2} (b_i b_j^{\varepsilon_j} b_k^{\varepsilon_k} b_4 - 1) \\ &= (b_i b_4 - b_i^{-1} b_4^{-1})^{-2} \; \{F(b_i, b_4; b_j, b_k) - F(b_i, b_4^{-1}; b_j, b_k)\} \\ &\qquad \{F(b_i, b_4; b_j, b_k) - F(b_j, b_k \;; b_i, b_4)\} \\ &\qquad \{F(b_i, b_4; b_j, b_k) - F(b_j, b_k^{-1}; b_i, b_4)\}. \end{aligned} \tag{42}$$

label	fixed point	existence	smoothness condition
1	$P(b_i, b_4; b_j, b_k)$	$\kappa_i + \kappa_4 \not\in \mathbb{Z}$	$\kappa_i \not\in \mathbb{Z}, \quad \kappa_4 \not\in \mathbb{Z}, \quad \kappa_i + \kappa_4 \pm \kappa_j \pm \kappa_k \not\in 2\mathbb{Z}+1$
2	$P(b_i, b_4^{-1}; b_j, b_k)$	$\kappa_i - \kappa_4 \not\in \mathbb{Z}$	$\kappa_i \not\in \mathbb{Z}, \quad \kappa_4 \not\in \mathbb{Z}, \quad \kappa_i - \kappa_4 \pm \kappa_j \pm \kappa_k \not\in 2\mathbb{Z}+1$
3	$P(b_j, b_k; b_i, b_4)$	$\kappa_j + \kappa_k \not\in \mathbb{Z}$	$\kappa_j \not\in \mathbb{Z}, \quad \kappa_k \not\in \mathbb{Z}, \quad \kappa_j + \kappa_k \pm \kappa_i \pm \kappa_4 \not\in 2\mathbb{Z}+1$
4	$P(b_j, b_k^{-1}; b_i, b_4)$	$\kappa_j - \kappa_k \not\in \mathbb{Z}$	$\kappa_j \not\in \mathbb{Z}, \quad \kappa_k \not\in \mathbb{Z}, \quad \kappa_j - \kappa_k \pm \kappa_i \pm \kappa_4 \not\in 2\mathbb{Z}+1$

Table 3: Smooth fixed points $x \in \mathrm{Fix}_j^\circ(\theta)$ with nonzero $x_i^2 - 4$

Hence $P(b_i, b_4; b_j, b_k)$ is a smooth point of $\mathcal{S}(\theta)$ if and only if $F(b_i, b_4; b_j, b_k)$ is a simple root of equation (39). In terms of $\kappa \in \mathcal{K}$, the existence and smoothness conditions for $P(b_i, b_4; b_j, b_k)$ are given by $\kappa_i + \kappa_4 \not\in \mathbb{Z}$ and $\kappa_i \not\in \mathbb{Z}$, $\kappa_4 \not\in \mathbb{Z}$, $\kappa_i + \kappa_4 \pm \kappa_j \pm \kappa_k \not\in 2\mathbb{Z}+1$, respectively.

Lemma 6.1 *The smooth fixed points $x \in \mathrm{Fix}_j^\circ(\theta)$ with nonzero $x_i^2 - 4$ are precisely those points in Table 3 which satisfy the existence and smoothness conditions mentioned there.*

The fixed points in Table 3 is closely related to the configuration of lines on the affine cubic surface $\mathcal{S}(\theta)$ or on its compactification $\overline{\mathcal{S}}(\theta)$ by the standard embedding

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}}(\theta) \subset \mathbb{P}^3, \qquad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3],$$

where the projective cubic surface $\overline{\mathcal{S}}(\theta)$ is defined by the homogeneous equation

$$F(X, \theta) := X_1X_2X_3 + X_0(X_1^2 + X_2^2 + X_3^2) - X_0^2(\theta_1X_1 + \theta_2X_2 + \theta_3X_3) + \theta_4X_0^3 = 0.$$

It is obtained from the affine surface $\mathcal{S}(\theta)$ by adding three lines at infinity

$$L_i = \{\, X \in \mathbb{P}^3 \;:\; X_0 = X_i = 0 \,\} \qquad (i = 1, 2, 3),$$

whose union $L = L_1 \cup L_2 \cup L_3$ is called the tritangent lines at infinity.

It is well known that a smooth projective cubic surface has exactly 27 lines on it. We describe them in the current situation [20]. Let $L_i(b_i, b_4; b_j, b_k)$ be the line in $\mathbb{P}^3$ defined by

$$X_i = (b_ib_4 + b_i^{-1}b_4^{-1})X_0, \qquad X_j + (b_ib_4)X_k = \{b_i(b_k + b_k^{-1})\} + b_4(b_j + b_j^{-1})X_0. \qquad (43)$$

1	$L_{i1}^+ = L_i(b_i, b_4; b_j, b_k)$	$L_{i1}^- = L_i(b_i^{-1}, b_4^{-1}; b_j, b_k)$
2	$L_{i2}^+ = L_i(b_i, b_4^{-1}; b_j, b_k)$	$L_{i2}^- = L_i(b_i^{-1}, b_4; b_j, b_k)$
3	$L_{i3}^+ = L_i(b_j, b_k; b_i, b_4)$	$L_{i3}^- = L_i(b_j^{-1}, b_k^{-1}; b_i, b_4)$
4	$L_{i4}^+ = L_i(b_j, b_k^{-1}; b_i, b_4)$	$L_{i4}^- = L_i(b_j^{-1}, b_k; b_i, b_4)$

Table 4: Eight lines intersecting the line L_i at infinity, divided into four pairs

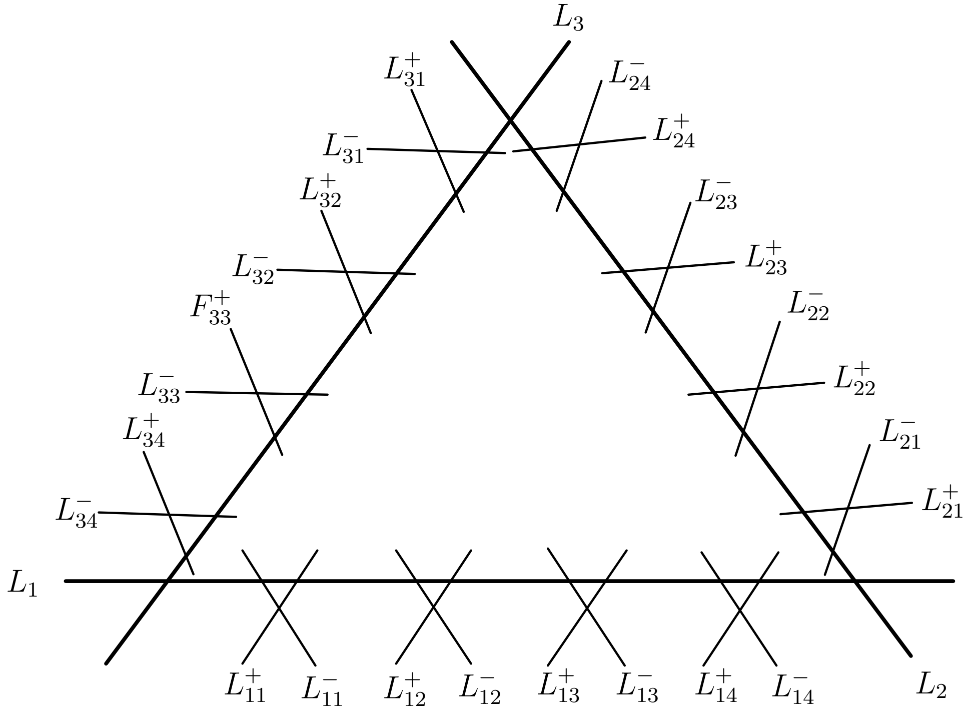


Figure 11: The 27 lines on a smooth cubic surface viewed from the tritangent lines at infinity

For each $i \in \{1,2,3\}$ the eight lines in Table 4 are the only lines on $\overline{\mathcal{S}}(\theta)$ that intersect the i-th line L_i at infinity, but they do not intersect the remaining two lines L_j and L_k at infinity. These lines are divided into four pairs as in Table 4. The surface $\overline{\mathcal{S}}(\theta)$ is always smooth at infinity [20] and hence, if $\kappa \in \mathcal{K} - \mathbf{Wall}$, then $\overline{\mathcal{S}}(\theta)$ is smooth everywhere. In this case, the two lines in the same pair intersect, while two lines from different pairs do not. The intersection point of the i-th pair is exactly the i-th fixed point in Table 3. See Figure 11 for a total image of these situations. *Caution*: for a pair of distinct indices i and j, the intersection relations between $L_{i\mu}^{\pm}$ and $L_{j\nu}^{\pm}$ are not depicted in the Figure 11. We also remark that in some degenerate cases the lines $L_{i\mu}^{+}$ and $L_{i\mu}^{-}$ may meet in a point on the line L_i at infinity.

Next we consider the case where the determinant $4 - x_i^2$ of system (37) vanishes. In other words we ask when the fixed point set $\mathrm{Fix}_j(\theta)$ contains points x such that $x_i \in \{\pm 2\}$.

Lemma 6.2 $\mathrm{Fix}_j(\theta)$ *contains a point x such that $x_i = 2\delta$ with $\delta \in \{\pm 1\}$ if and only if either*

(1) $b_i b_4 = b_i b_4^{-1} = \delta$; *or*

(2) $b_j b_k = b_j b_k^{-1} = \delta$; *or*

(3) $b_i b_4^{\varepsilon_4} = b_j b_k^{\varepsilon_k} = \delta$ *for some double sign* $(\varepsilon_k, \varepsilon_4) \in \{\pm 1\}^2$.

If this is the case, then $\theta_k = \delta\theta_j$ and all such poins x are exactly those points on the line

$$\ell_j^{\delta} := \{\, x_i = 2\delta,\ x_j + \delta x_k = \theta_j/2 \,\}. \tag{44}$$

In particular $\ell_j^{\delta} \subset \mathrm{Fix}_j(\theta)$ precisely when $x_i = 2\delta$ is a multiple root of the quartic equation (39).

$x_i \in \{\pm 2\}$	multiplicity	component	remark
no	simple	smooth point	intersection point of $L_{i\mu}^{\pm}$
no	multiple	singular point	Riccati locus
yes	multiple	line ℓ_j^+ or ℓ_j^-	line contains singular points
yes	simple	empty	$L_{i\mu}^{\pm}$ intersects at infinity

Table 5: The roots of quartic equation (39) and the components of $\mathrm{Fix}_j(\theta)$

Proof. If $x_i = 2\delta$ with $\delta \in \{\pm 1\}$ then system (37) is linearly dependent, so that $\theta_k - \delta\theta_j = 0$. However, since $\theta_k - \delta\theta_j = (b_i b_j)^{-1}(b_i b_4 - \delta)(b_i b_4^{-1} - \delta)(b_j b_k - \delta)(b_j b_k^{-1} - \delta)$, we have either $b_i b_4^{\varepsilon_4} = \delta$ for some sign $\varepsilon_4 \in \{\pm 1\}$ or $b_j b_k^{\varepsilon_k} = \delta$ for some sign $\varepsilon_k \in \{\pm 1\}$. Taking the equation $x_j + \delta x_k = \theta_j/2$ into account, we observe that $f(x, \theta)$ factors as

$$f(x,\theta) = \begin{cases} -(2b_i b_j)^{-2}(b_i b_4^{-\varepsilon_4} - \delta)^2 (b_j b_k - \delta)^2 (b_j b_k^{-1} - \delta)^2 & (\text{if} \quad b_i b_4^{\varepsilon_4} = \delta), \\ -(2b_i b_j)^{-2}(b_j b_k^{-\varepsilon_k} - \delta)^2 (b_i b_4 - \delta)^2 (b_i b_4^{-1} - \delta)^2 & (\text{if} \quad b_j b_k^{\varepsilon_k} = \delta). \end{cases}$$

If $b_i b_4^{\varepsilon_4} = \delta$ then equation $f(x,\theta) = 0$ yields either $b_i b_4^{-\varepsilon_4} = \delta$ or $b_j b_k^{\varepsilon_k} = \delta$ for some sign $\varepsilon_k \in \{\pm 1\}$; the former case falls into case (1) while the latter falls into case (3). In a similar manner the other case $b_j b_k^{\varepsilon_k} = \delta$ falls into case (2) or case (3).

Next, if $\mathrm{Fix}_j(\theta)$ contains the line ℓ_j^{δ}, then what we have just proved implies that

$$\begin{array}{rclcll} F(b_i, b_4; b_j, b_k) & = & F(b_i, b_4^{-1}; b_j, b_k) & = & 2\delta & \text{if condition (1) is satisfied;} \\ F(b_j, b_k; b_i, b_4) & = & F(b_j, b_k^{-1}; b_i, b_4) & = & 2\delta & \text{if condition (2) is satisfied;} \\ F(b_i, b_4^{\varepsilon_4}; b_j, b_k) & = & F(b_j, b_k^{\varepsilon_k}; b_i, b_4) & = & 2\delta & \text{if condition (3) is satisfied.} \end{array}$$

Hence $x_i = 2\delta$ is a multiple root of the quartic equation (39). Conversely, if $x_i = 2\delta$ is a multiple root of (39), then we can trace the argument backwards to conclude that the system (23) admits the line solution ℓ_j^{δ}, that is, $\mathrm{Fix}_j(\theta)$ contains ℓ_j^{δ}. $\square$

Summarizing the arguments so far yields a classification of the irreducible components of the algebraic set $\mathrm{Fix}_j(\theta)$ in terms of certain roots of quartic equation (39).

Theorem 6.3 *Any irreducible component of* $\mathrm{Fix}_j(\theta)$ *is just a single point or a single affine line; the former is called a point component and the latter is called a line component respectively. The irreducible components of* $\mathrm{Fix}_j(\theta)$ *are in one-to-one correspondence with those roots of quartic equation (39) which are not a simple root* $x = (x_1, x_2, x_3)$ *such that* $x_i \in \{\pm 2\}$.

(1) *A simple root with* $x_i \notin \{\pm 2\}$ *corresponds to a point component that is a smooth point of the surface* $\mathcal{S}(\theta)$ *and is given in Table 3.*

(2) *A multiple root with* $x_i \notin \{\pm 2\}$ *corresponds to a point component that is a singular point of the surface* $\mathcal{S}(\theta)$ *and is associated with Riccati solutions.*

(3) *A multiple root with* $x_i \in \{\pm 2\}$ *corresponds to a line component; either* ℓ_j^+ *or* ℓ_j^-.

(4) *A simple root with* $x_i \in \{\pm 2\}$ *corresponds to no component of* $\mathrm{Fix}_j(\theta)$.

A summary of Theorem 6.3 is given in Table 5 and the following remark may be helpful.

Remark 6.4 The assertions (3) and (4) of Theorem 6.3 may be well understood through the degeneration of line configration on the projective surface $\overline{\mathcal{S}}(\theta)$ as the parameter $\theta = \mathrm{rh}(\kappa)$ tends to a special position. For a generic value of θ the lines $L^{\pm}_{i\mu}$ intersect in a single (smooth) point on the affine part $\mathcal{S}(\theta)$ of $\overline{\mathcal{S}}(\theta)$. If the parameter θ tends to a special position so that a corresponding root x_i of quartic equation (39) approaches $\{\pm 2\}$, then the two line $L^{\pm}_{i\mu}$ are getting "parallel" and eventually either coincide completely or meet in a point at infinity. The former case falls into assertion (3) and the latter case falls into assertion (4) respectively.

Let us investigate more closely the case where $\mathrm{Fix}_j(\theta)$ contains line components.

Lemma 6.5 *Let* $\theta = \mathrm{rh}(\kappa)$ *with* $\kappa \in \mathcal{K}$ *and* (i,j,k) *be any cyclic permutation of* $(1,2,3)$.

(1) $\mathrm{Fix}_j(\theta)$ *contains either* ℓ_j^+ *or* ℓ_j^- *but not both of them if and only if* κ *lies in a* $\widetilde{W}(D_4^{(1)})$*-stratum appearing in the following adjacency diagram* (*see also Figure 10*) :

$$\begin{array}{ccc} (A_1^{\oplus 2})_i & \longrightarrow & A_1^{\oplus 3} \\ \big\downarrow & & \big\downarrow \\ (A_3)_i & \longrightarrow & D_4 \end{array} \qquad (45)$$

(2) $\mathrm{Fix}_j(\theta)$ *contains both* ℓ_j^+ *and* ℓ_j^- *if and only if* $\theta = (0,0,0,-4)$, *that is, precisely when* κ *is in the* $\widetilde{W}(D_4^{(1)})$*-stratum of type* $A_1^{\oplus 4}$. *In this case one has* $\mathrm{Fix}_j(\theta) = \ell_j^+ \amalg \ell_j^-$.

Proof. Lemma 6.2 implies that $\mathrm{Fix}_j(\theta)$ contains at least one of ℓ_j^+ and ℓ_j^- if and only if either (a) $b_i b_4 = b_i b_4^{-1} \in \{\pm 1\}$; or (b) $b_j b_k = b_j b_k^{-1} \in \{\pm 1\}$; or (c) $b_i b_4^{\varepsilon_4} = b_j b_k^{\varepsilon_k} \in \{\pm 1\}$ for some double sign $(\varepsilon_k, \varepsilon_4) \in \{\pm 1\}^2$. This property is invariant under the action of $\widetilde{W}(D_4^{(1)}) = \mathrm{Kl} \ltimes W(D_4^{(1)})$ on $\mathcal{K}$. Using this action we can reduce conditions (a) and (c) to condition (b). First, observe that the permutation $(i,j)(k4) \in \mathrm{Kl}$ induces the map $(b_0, b_i, b_j, b_k, b_4) \mapsto (b_0, b_j, b_i, -b_4, -b_k)$, which reduces condition (a) to condition (b). Next, formula (13) implies that the reflection w_i induces the multiplicative transformation $w_i : B \to B$, $b \mapsto b'$, where

$$b'_j = \begin{cases} -b_j b_i^{c_{ij}} & (i = 4, j = 0), \\ b_j b_i^{c_{ij}} & (\text{otherwise}). \end{cases}$$

Applying w_4 or w_k if necessary, we may assume from the beginning that $\varepsilon_4 = 1$ and $\varepsilon_k = -1$ in condition (c). Then using w_0 there yields $b_0^2 b_i b_4 = b_j b_k^{-1} \in \{\pm 1\}$. But since $b_0^2 b_i b_4 b_j b_k = 1$, we have $b_j b_k = b_j b_k^{-1} \in \{\pm 1\}$, that is, condition (b). Note that condition (b) means κ_j, $\kappa_k \in \mathbb{Z}$. On the other hand, the extended affine Weyl group $\widetilde{W}(D_4^{(1)})$ contains shifts

$$\begin{aligned} (\kappa_0, \kappa_i, \kappa_j, \kappa_k, \kappa_4) &\mapsto (\kappa_0, \kappa_i - 1, \kappa_j + 1, \kappa_k, \kappa_4), \\ (\kappa_0, \kappa_i, \kappa_j, \kappa_k, \kappa_4) &\mapsto (\kappa_0, \kappa_i, \kappa_j, \kappa_k + 1, \kappa_4 - 1). \end{aligned} \qquad (46)$$

Repeated applications of these operations and their inverses can shift κ_j and κ_k independently by arbitrary integers. Thus the condition κ_j, $\kappa_k \in \mathbb{Z}$ can further be reduced to $\kappa_j = \kappa_k = 0$.

Thus we have shown that if $\mathrm{Fix}_j(\theta)$ contains at least one of ℓ_j^+ and ℓ_j^-, then κ must lie in the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_1^{\oplus 2})_i$ or on its boundary strata of types $(A_3)_i$, $A_1^{\oplus 3}$, D_4, $A_1^{\oplus 4}$. Moreover, it is easy to see that the converse is also true.

For a sign $\delta \in \{\pm 1\}$ the conditions (1), (2), (3) in Lemma 6.2 are denoted by (1^δ), (2^δ), (3^δ), respectively. Now we assume that $\mathrm{Fix}_j(\theta)$ contains both ℓ_j^+ and ℓ_j^-. Then there exists a pair of conditions, one from $\{(1^+), (2^+), (3^+)\}$ and the other from $\{(1^-), (2^-), (3^-)\}$, that are valid at the same time. Such a pair can be consistent only if it is either $(1^-) + (2^-)$; or $(2^+) + (1^-)$; or $(3^+) + (3^-)$ where if the sign for (3^+) is $(\varepsilon_k, \varepsilon_4)$ then the sign for (3^-) must be its antipode $(-\varepsilon_k, -\varepsilon_4)$. The first and second pairs lead to $b_1^2 = b_2^2 = b_3^2 = b_4^2 = 1$ and to $b_1 b_2 b_3 b_4 = -1$, while the third pair yields $b_1^2 = b_2^2 = b_3^2 = b_4^2 = -1$. These are nothing but the conditions (a) and (b) in Example 3.4. (1). Therefore κ must lie in the stratum of type $A_1^{\oplus 4}$. Combining this with the discussion in the last paragraph establishes the assertion (1), as well as a large part of the assertion (2). The only thing yet to be proved is the assertion that if $\mathrm{Fix}_j(\theta)$ contains both ℓ_j^+ and ℓ_j^-, then $\mathrm{Fix}_j(\theta) = \ell_j^+ \amalg \ell_j^-$. For this, the last part of Lemma 6.2 implies that both $x_i = 2$ and $x_i = -2$ are multiple roots of the quartic equation (39), so that there are no other roots of the equation (39). Thus $\mathrm{Fix}_j(\theta)$ has no elements other than those in $\ell_j^+ \amalg \ell_j^-$. $\square$

Now we turn our attention to periodic points and investigate the set $\widetilde{\mathrm{Per}}{}^{\circ}_j(\theta; n)$ of periodic points of prime period $n > 1$ on the non-Riccati locus.

Lemma 6.6 *For any integer $n > 1$ the set $\widetilde{\mathrm{Per}}{}^{\circ}_j(\theta; n)$ is biholomorphic to the disjoint union of $\varphi(n)$ copies of $\mathbb{C}^\times$, where $\varphi(n)$ denotes the number of integers $0 < m < n$ coprime to n.*

Proof. By Lemma 4.5 we can identify $\widetilde{\mathrm{Per}}{}^{\circ}_j(\theta; n)$ with $\mathrm{Per}_j(\theta; n)$ and hence may work downstairs. For any integer $0 < m < n$ coprime to n, we consider the projective curve $\overline{C}_m$ in $\mathbb{P}^3$ defined by

$$\begin{aligned} \{4\cos^2(\pi m/n) - 2\theta_i \cos(\pi m/n) + \theta_4\} X_0^2 - X_0(\theta_j X_j + \theta_k X_k) & \\ + X_j^2 + X_k^2 + 2\cos(\pi m/n) X_j X_k &= 0, \qquad (47) \\ X_i - 2\cos(\pi m/n) X_0 &= 0, \qquad (48) \end{aligned}$$

where (47) is obtained from $F(X, \theta) = 0$ by substituting (48) and factoring X_0 out of it. It follows from $-2 < 2\cos(\pi m/n) < 2$ that $\overline{C}_m$ is an irreducible smooth conic curve. By equations (24) of Lemma 4.4 the closure $\overline{\mathrm{Per}}_j(\theta; n)$ of $\mathrm{Per}_j(\theta; n)$ in $\overline{\mathcal{S}}(\theta)$ is the union of these $\varphi(n)$ curves $\overline{C}_m$. The curve $\overline{C}_m$ intersects the lines $L = L_i \cup L_j \cup L_k$ at infinity in the two points

$$P_m^{\pm}: \quad [X_0 : X_i : X_j : X_k] = [0 : 0 : -1 : \exp(\pm \pi\sqrt{-1} m/n)] \in L_i.$$

If $C_m := \overline{C}_m - \{P_m^+, P_m^-\}$, then C_m is biholomorphic to $\mathbb{C}^\times$, since $\overline{C}_m \cong \mathbb{P}^1$. So $\mathrm{Per}_j(\theta; n)$ is the disjoint union of the $\varphi(n)$ curves C_m with $0 < m < n$, $(m, n) = 1$, and hence biholomorphic to the disjoint union of $\varphi(n)$ copies of $\mathbb{C}^\times$. $\square$

7 Case-by-Case Study

We make case-by-case studies of $\widetilde{\mathrm{Fix}}_j(\theta)$ and $\widetilde{\mathrm{Per}}{}^{e}_j(\theta; n)$ according to the adjacency diagram in Figure 10. Now we need to introduce some notation. Recall that we have the resolution of

singularities (15) which restricts to an isomorphism $\varphi : \widetilde{\mathcal{S}}^\circ(\theta) \to \mathcal{S}^\circ(\theta)$ and that the smooth fixed points $\mathrm{Fix}_j^\circ(\theta)$ in $\mathcal{S}^\circ(\theta)$ are listed in Table 3. For each $P \in \mathrm{Fix}_j^\circ(\theta)$ let $\widetilde{P} \in \overset{\circ}{\widetilde{\mathrm{Fix}}}_j(\theta)$ denote its lift through the isomorphism φ. For example, $\widetilde{P}(b_i, b_4; b_j, b_k)$ denotes the lift of $P(b_i, b_4; b_j, b_k)$. If $\{\cdots\}$ is a set of expressions $\widetilde{P}(b_i, b_4^{\pm 1}; b_j, b_k)$, $\widetilde{P}(b_j, b_k^{\pm 1}; b_i, b_4)$, then we denote by $\{\{\cdots\}\}$ its subset obtained by discarding those expressions which do not satisfy either the existence condition or the smoothness condition of Table 3. An example is given in (49) below.

Example 7.1 ($\emptyset$) Consider the $\widetilde{W}(D_4^{(1)})$-stratum of type $\emptyset$, namely, the big open $\mathcal{K} - \mathbf{Wall}$.

$$\widetilde{\mathrm{Fix}}_j(\theta) = \{\{\, \widetilde{P}(b_i, b_4; b_j, b_k),\ \widetilde{P}(b_i, b_4^{-1}; b_j, b_k),\ \widetilde{P}(b_j, b_k; b_i, b_4),\ \widetilde{P}(b_j, b_k^{-1}; b_i, b_4) \,\}\}. \tag{49}$$

Here we have only to care the existence condition, as we are in the big open where the smoothness condition is fulfilled by hypothesis. If a finer stratification of $\mathcal{K}$ attached to the $W(F_4^{(1)})$-action on $\mathcal{K}$ is introduced, then a more precise description of (49) is feasible, detecting how many and which elements are there in (49), but the details are omitted. We only remark that $\widetilde{\mathrm{Fix}}_j(\theta)$ consists of four distinct points in the most generic case where none of $\kappa_i \pm \kappa_4$ and $\kappa_j \pm \kappa_k$ are integers. As for the periodic points, since there is no Riccati locus, we have

$$\widetilde{\mathrm{Per}}_j(\theta; n) = \overset{\circ}{\widetilde{\mathrm{Per}}}_j(\theta; n), \qquad \overset{e}{\widetilde{\mathrm{Per}}}_j(\theta; n) = \emptyset, \qquad (n > 1).$$

Example 7.2 (A_1) Consider the $\widetilde{W}(D_4^{(1)})$-stratum of type A_1. We may assume that $\kappa_0 = 0$ so that $b_0 = 1$ and $b_i b_j b_k b_4 = 1$. Note that none of b_i^2, b_j^2, b_k^2, b_4^2 equals 1. We claim that $b_j^2 b_k^2 \neq 1$. Otherwise, we would have $\kappa_j + \kappa_k \in \mathbb{Z}$. Applying a shift as in (46) to κ repeatedly, we may assume that $\kappa_j + \kappa_k = 0$ while keeping the condition $\kappa_0 = 0$. Then the transformation $w_0 w_j$ sends κ to κ' with $\kappa_j' = 0$ and $\kappa_k' = \kappa_j + \kappa_k = 0$, so that one has $\kappa \in \overline{\mathcal{K}}_{\{j,k\}}$, namely, κ lies in the closure of the stratum of type $(A_1^{\oplus 2})_i$. This contradicts the assumption that we are in the stratum of type A_1. In this case, the surface $\mathcal{S}(\theta)$ has a unique singular point of type A_1 at

$$(x_i, x_j, x_k) = (b_i b_4 + b_i^{-1} b_4^{-1}, b_j b_4 + b_j^{-1} b_4^{-1}, b_k b_4 + b_k^{-1} b_4^{-1}).$$

Blow up $\mathcal{S}(\theta)$ at this point to obtain a minimal resolution (15). Write the blowing-up as

$$(x_i, x_j, x_k) = (u_i u_j + b_i b_4 + b_i^{-1} b_4^{-1}, u_j + b_j b_4 + b_j^{-1} b_4^{-1}, u_k u_j + b_k b_4 + b_k^{-1} b_4^{-1})$$

in terms of coordinates (u_i, u_j, u_k). The exceptional set e is the irreducible quadratic curve

$$u_j = b_i b_j b_k + (1 + b_i^2 b_j^2) b_k u_i + (1 + b_k^2 b_j^2) b_i u_k + b_i b_j b_k (u_i^2 + u_k^2) + (1 + b_i^2 b_k^2) b_j u_i u_k = 0,$$

which can be paramatrized as $u_j = 0$ and

$$\begin{aligned} u_i &= \frac{b_i^2 b_4^2 (b_j^2 b_k^2 - 1)^3 t}{b_j \{b_k (b_i^2 - 1)(b_j^2 - 1) + b_i (b_j^2 b_k^2 - 1) t\}\{(b_k^2 - 1)(b_4^2 - 1) + b_i b_k b_4^2 (b_j^2 b_k^2 - 1) t\}}, \\ u_k &= \frac{\{b_j^2 b_k (b_i^2 - 1)(b_k^2 - 1) - b_i (b_j^2 b_k^2 - 1) t\}\{b_k (b_j^2 - 1)(1 - b_4^2) + b_i b_4^2 (b_j^2 b_k^2 - 1) t\}}{b_j \{b_k (b_i^2 - 1)(b_j^2 - 1) + b_i (b_j^2 b_k^2 - 1) t\}\{(b_k^2 - 1)(b_4^2 - 1) + b_i b_k b_4^2 (b_j^2 b_k^2 - 1) t\}}. \end{aligned}$$

In terms of this parametrization, the lifted transformation $\tilde{g}_j^2$ acts on the exceptional curve $e \simeq \mathbb{P}^1$ by the multiplication $t \mapsto b_j^2 b_k^2 t$. Since $b_j^2 b_k^2 \neq 1$, the set $\overset{e}{\widetilde{\mathrm{Fix}}}_j(\theta)$ consists of the two

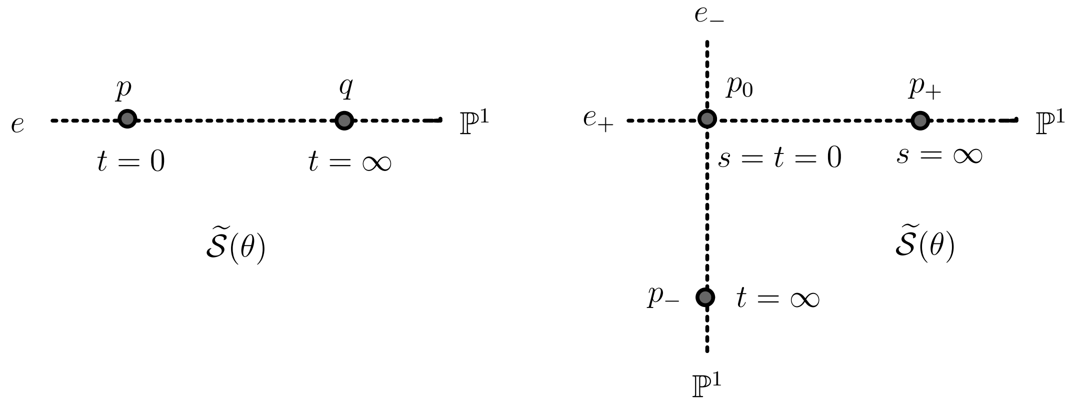


Figure 12: Surface of types A_1 (left) and A_2 (right)

points, say p and q, corresponding to $t=0$ and $t=\infty$ (see Figure 12, left). On the other hand, the possible candidates for the smooth fixed points $\widetilde{\mathrm{Fix}}^{\circ}_j(\theta)$ are only the points of labels 2 and 4 in Table 3, since those of labels 1 and 3 do not satisfy the smoothness condition. Thus,

$$\widetilde{\mathrm{Fix}}_j(\theta) = \{\{\widetilde{P}(b_i, b_4^{-1}; b_j, b_k),\ \widetilde{P}(b_j, b_k^{-1}; b_i, b_4)\}\} \amalg \{p, q\}. \tag{50}$$

As for the Riccati periodic points $\widetilde{\mathrm{Per}}^{e}_j(\theta; n)$, the discussion above implies that for any $n > 1$,

$$\widetilde{\mathrm{Per}}^{e}_j(\theta; n) = \begin{cases} e & \text{(if } b_j b_k \text{ is a primitive } 2n\text{-th root of unity),} \\ \emptyset & \text{(otherwise).} \end{cases}$$

Example 7.3 (A_2) Consider the $\widetilde{W}(D_4^{(1)})$-stratum of type A_2. We may assume that $\kappa_0 = \kappa_i = 0$ so that $b_0 = b_i = 1$. Then the surface $\mathcal{S}(\theta)$ has a unique singular point of type A_2 at

$$(x_i, x_j, x_k) = (b_4 + b_4^{-1}, b_j b_4 + b_j^{-1} b_4^{-1}, b_k b_4 + b_k^{-1} b_4^{-1}).$$

Blow up $\mathcal{S}(\theta)$ at this point to obtain a minimal resolution (15). Write the blowing-up as

$$(x_i, x_j, x_k) = (u_i u_j + b_4 + b_4^{-1}, u_j + b_j b_4 + b_j^{-1} b_4^{-1}, u_k u_j + b_k b_4 + b_k^{-1} b_4^{-1})$$

in terms of coordinates (u_i, u_j, u_k). The exceptional set e is the union of two lines

$$e^+ \ :\quad u_j = b_k u_i + u_k + b_j b_k = 0, \qquad e^- \ :\quad u_j = b_k^{-1} u_i + u_k + b_j^{-1} b_k^{-1} = 0,$$

intersecting in a point. These lines are parametrized as

$$e^+ \ :\quad (u_i, u_j, u_k) = \left(\frac{b_j^2 b_k^2 - 1}{b_j(1-b_k^2) + (b_j^2 b_k^2 - 1)s}, 0, \frac{b_k(1-b_j^2) + b_j b_k (1 - b_j^2 b_k^2) s}{b_j(1-b_k^2) + (b_j^2 b_k^2 - 1)s} \right),$$

$$e^- \ :\quad (u_i, u_j, u_k) = \left(\frac{b_j^2 b_k^2 - 1}{b_j(1-b_k^2) + (b_j^2 b_k^2 - 1)t}, 0, \frac{b_k(1-b_j^2) + b_j^{-1} b_k^{-1} (1 - b_j^2 b_k^2) t}{b_j(1-b_k^2) + (b_j^2 b_k^2 - 1)t} \right),$$

with the intersection point corresponding to $s=t=0$. In terms of these parametrizations, the lifted transformation $\tilde{g}_j^2$ acts on e^+ and e^- by the multiplications $s \mapsto b_j^{-2} b_k^{-2} s$ and $t \mapsto b_j^2 b_k^2 t$,

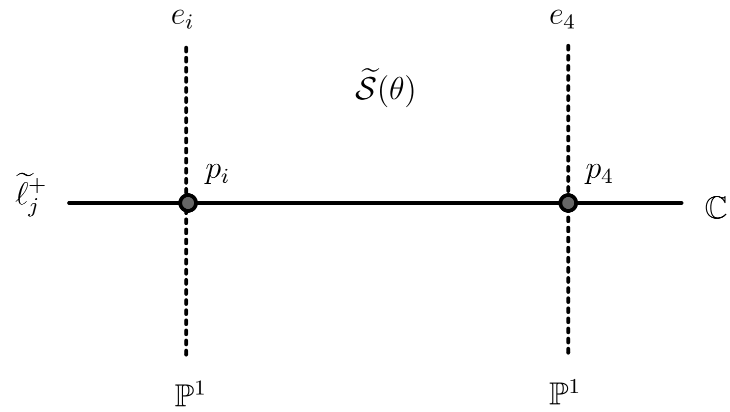


Figure 13: Surface of type $(A_1^{\oplus 2})_i$

which are rewritten as $s \mapsto b_4^2 s$ and $t \mapsto b_4^{-2} t$, since $b_j b_k b_4 = 1$. Note that $b_4^2 \neq 1$, for otherwise κ would be in the closure of the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_3)_i$. So $\tilde{g}_j^2$ has exactly two fixed points p_0 and p_+ on e^+ corresponding to $s = 0$ and $s = \infty$. Similarly $\tilde{g}_j^2$ has exactly two fixed points p_0 and p_- on e^- corresponding to $t = 0$ and $t = \infty$, where p_0 is the intersection point of e^+ and e^- (see Figure 12, right). Thus we have $\widetilde{\mathrm{Fix}}_j^{e}(\theta) = \{p_0, p_+, p_-\}$. Next we consider the smooth fixed point of $\tilde{g}_j^2$ on $\widetilde{\mathcal{S}}(\theta)$. Since we are assuming that $\kappa_i = \kappa_j + \kappa_k + \kappa_4 = 1$, the points of labels 1, 2, 3 in Table 3 do not satisfy the smoothness condition and that of label 4 is the only smooth fixed point. Thus $\widetilde{\mathrm{Fix}}_j^{\circ}(\theta) = \{\widetilde{P}(b_j, b_k^{-1}; b_i, b_4)\}$ and hence

$$\widetilde{\mathrm{Fix}}_j(\theta) = \{\widetilde{P}(b_j, b_k^{-1}; b_i, b_4),\ p_0,\ p_+,\ p_-\}. \tag{51}$$

In the remaining cases presented below, $\mathrm{Fix}_j(\theta)$ contains at least one line component.

Example 7.4 ($\boldsymbol{A_1^{\oplus 2}}$) First we consider $\widetilde{\mathrm{Fix}}_j(\theta)$ and $\widetilde{\mathrm{Per}}_j^{e}(\theta)$ on the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_1^{\oplus 2})_i$. We may assume that $\kappa_j = \kappa_k = 0$ so that $b_j = b_k = 1$. Since our stratum is not of type $(A_3)_i$ nor of type D_4, we have $(b_i b_4 - 1)(b_i b_4^{-1} - 1) \neq 0$ or equivalently $b_i + b_i^{-1} \neq b_4 + b_4^{-1}$. In this case $\mathrm{Fix}_j(\theta)$ contains the line ℓ_j^+ but does not the line ℓ_j^- and the surface $\mathcal{S}(\theta)$ has two singular points of type A_1 at $(x_i, x_j, x_k) = (2, b_i + b_i^{-1}, b_4 + b_4^{-1})$ and $(x_i, x_j, x_k) = (2, b_4 + b_4^{-1}, b_i + b_i^{-1})$. We denote the former singularity by q_i and the latter by q_4 respectively; both singularities lie on the line ℓ_j^+. Blow up $\mathcal{S}(\theta)$ at these points to obtain a minimal resolution as in (15). Let $\widetilde{\ell}_j^+$ be the strict transform of ℓ_j^+, and let e_i and e_4 be the exceptional curves over q_i and q_4 respectively. Moreover let p_i be the intersection point of $\widetilde{\ell}_j^+$ and e_i. Similarly let p_4 be the intersection point of $\widetilde{\ell}_j^+$ and e_4 (see Figure 13). Then the blowing-up at the point q_i is represented as

$$(x_i, x_j, x_k) = (u_i u_j + 2, u_j + b_i + b_i^{-1}, u_k u_j + b_4 + b_4^{-1})$$

in terms of coordinates (u_i, u_j, u_k) around $(0, 0, 0)$. The strict transform $\widetilde{\ell}_j^+$ and the exceptional curve e_i are given by $u_i = u_k + 1 = 0$ and

$$u_j = (b_i b_4)(u_i^2 + u_k^2) + (b_i^2 + 1) b_4 (u_i u_k) + b_i (b_4^2 + 1) u_i + 2(b_i b_4) u_k + (b_i b_4) = 0.$$

The exceptional curve e_i admits a parametrization

$$u_i = \frac{(b_i b_4 - 1)(b_i b_4^{-1} - 1)}{(t + b_i)(b_i t + 1)}, \qquad u_j = 0, \qquad u_k = -\frac{b_i (t + b_4)(b_4 t + 1)}{b_4 (t + b_i)(b_i t + 1)}, \tag{52}$$

where the intersection point p_i has coordinates $(u_i, u_j, u_k) = (0, 0, -1)$, which corresponds to $t = \infty$. The lifted transformation $\tilde{g}_j^2$ acts on e_i as a Möbius transformation fixing p_i. Some computations show that in terms of the variable t this transformation is just the shift

$$t \mapsto t + (b_i + b_i^{-1}) - (b_4 + b_4^{-1}).$$

and hence a parabolic transformation. Thus $\tilde{g}_j^2$ has no periodic points on e_i other than the fixed point p_i. By symmetry, $\tilde{g}_j^2$ also acts on e_4 as a parabolic Möbius transformation fixing p_4 only. Summarizing the arguments, we conclude that on the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_1^{\oplus 2})_i$,

$$\widetilde{\mathrm{Fix}}_j(\theta) = \tilde{\ell}_j^+ \amalg \{\, \tilde{P}(b_i, b_4; b_j, b_k),\ \tilde{P}(b_i, b_4^{-1}; b_j, b_k) \,\}, \qquad \widetilde{\mathrm{Per}}_j^e(\theta; n) = \emptyset \quad (n > 1). \tag{53}$$

Next we consider $\widetilde{\mathrm{Fix}}_i(\theta)$ on the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_1^{\oplus 2})_i$. Some calculations show that there are parametrizations of e_i and e_4 such that $\tilde{g}_i^2$ acts on e_i and e_4 as the multiplications $t \mapsto b_4^2 t$ and $t \mapsto b_i^2 t$ respectively. (Modify (52) to get such parametrization.) Since $b_4^2 \neq 1$ and $b_i^2 \neq 1$, the transformation $\tilde{g}_i^2$ has exactly two fixed points, say p_{ii} and q_{ii}, on e_i, and exactly two fixed points, say p_{i4} and q_{i4}, on e_4. There are no smooth fixed points $\widetilde{\mathrm{Fix}}_i^{\circ}(\theta)$, because the smoothness condition of Table 3 with (i, j, k) replaced by (k, i, j) is not satisfied for any labels there. Thus we have $\widetilde{\mathrm{Fix}}_i(\theta) = \widetilde{\mathrm{Fix}}_i^e(\theta) = \{p_{ii}, q_{ii}, p_{i4}, q_{i4}\}$ and $\widetilde{\mathrm{Fix}}_i^{\circ}(\theta) = \emptyset$. By symmetry there is a similar characterization of $\widetilde{\mathrm{Fix}}_k(\theta)$. By permuting the indices (i, j, k), we have

$$\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\mathrm{Fix}}_j^e(\theta) = \{\text{four points}\}, \qquad \widetilde{\mathrm{Fix}}_j^{\circ}(\theta) = \emptyset, \tag{54}$$

on the $\widetilde{W}(D_4^{(1)})$-strata of types $(A_1^{\oplus 2})_j$ and $(A_1^{\oplus 2})_k$. A slightly further consideration yields

$$\widetilde{\mathrm{Per}}_j^e(\theta; n) = \begin{cases} e_j \amalg e_4 & \text{(if } b_j \text{ and } b_4 \text{ are primitive } 2n\text{-th roots of unity),} \\ e_j & \text{(if } b_4 \text{ is a primitive } 2n\text{-th root of unity, but } b_j \text{ is not),} \\ e_4 & \text{(if } b_j \text{ is a primitive } 2n\text{-th root of unity, but } b_4 \text{ is not),} \\ \emptyset & \text{(otherwise).} \end{cases}$$

on the stratum $(A_1^{\oplus 2})_j$ and a similar characterization of it on the stratum $(A_1^{\oplus 2})_k$.

Example 7.5 (A_3) First we consider $\widetilde{\mathrm{Fix}}_j(\theta)$ and $\widetilde{\mathrm{Per}}_j^e(\theta)$ on the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_3)_i$. We may assume that $\kappa_0 = \kappa_j = \kappa_k = 0$ and $\kappa_i + \kappa_4 = 1$ so that $b_j = b_k = 1$ and $b_i b_4 = 1$. But we have $b_i b_4^{-1} \not\in \{\pm 1\}$, since our stratum is not of type D_4. In this case $\mathrm{Fix}_j(\theta)$ contains the line ℓ_j^+ but does not the line ℓ_j^-. The surface $\mathcal{S}(\theta)$ has only one singular point of type A_3 at $(x_i, x_j, x_k) = (2, b_4 + b_4^{-1}, b_4 + b_4^{-1})$, which lies on the line ℓ_j^+. Blow up the singular point. This blowing-up is expressed as $(x_i, x_j, x_k) = (u_i u_j + 2, u_j + b_4 + b_4^{-1}, u_k u_j + b_4 + b_4^{-1})$ in terms of coordinates (u_i, u_j, u_k) around $(0, 0, 0)$. The strict transform of the surface $\mathcal{S}(\theta)$ is given by

$$u_j = b_4 u_i u_j u_k + b_4 u_i^2 + +b_4 u_k^2 + (b_4^2 + 1) u_i u_k + (b_4^2 + 1) u_i + 2 b_4 u_k + b_4 = 0,$$

which has yet one singular point, say q. The exceptional curve consists of two line components $u_j = u_i + b_4 u_k + b_4 = 0$ and $u_j = b_4 u_i + u_k + 1 = 0$, whose intersection point $(u_i, u_j, u_k) = (0, 0, -1)$ is exactly the singular point q. The strict transform of ℓ_j^+ is now given by $u_i = u_k + 1 = 0$, which

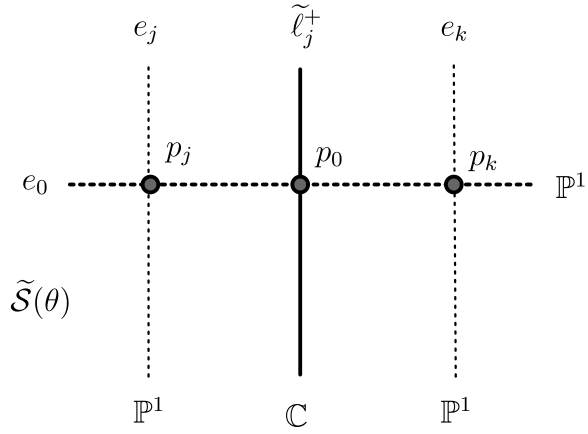


Figure 14: Surface of type $(A_3)_i$

also passes through q. Blow up again the singular point q. Let e_0 be the exceptional curve and let e_j, e_k, $\tilde{\ell}_j^+$ be the strict transforms of the lines $u_j = u_i + b_4 u_k + b_4 = 0$, $u_j = b_4 u_i + u_k + 1 = 0$, $u_i = u_k + 1 = 0$ respectively. If we express this blowing-up as $(u_i, u_j, u_k) = (v_i, v_i v_j, v_i v_k - 1)$, then the exceptional curve e_0 is given by $v_i = b_4 - b_4 v_j + (b_4^2 + 1) v_k + b_4 v_k^2 = 0$; e_j is given by $v_j = 1 + b_4 v_k = 0$; and e_k is given by $v_j = b_4 + v_k = 0$. The intersection point of e_0 and e_j is $(v_i, v_j, v_k) = (0, 0, -b_i)$ and that of e_0 and e_k is $(v_i, v_j, v_k) = (0, 0, -b_4)$. If e_j is parametrized as $(v_i, v_j, v_k) = ((t + b_i)^{-1}, 0, -b_i)$, then the transformation $\tilde{g}_j^2$ acts on e_j as the shift $t \mapsto t + b_4 - b_i$. Similarly, if e_k is parametrized as $(v_i, v_j, v_k) = ((t + b_4)^{-1}, 0, -b_4)$, then $\tilde{g}_j^2$ acts on e_k as the shift $t \mapsto t + b_4 - b_i$. Hence $\tilde{g}_j^2$ acts on e_j and e_k as parabolic Möbius transformations fixing only p_j and q_j. Then $\tilde{g}_j^2$ acts on e_0 as the identity, because it also fixes the intersection point p_0 of e_0 and $\tilde{\ell}_j^+$. Summarizing the arguments we see that on the stratum of type $(A_3)_i$,

$$\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\ell}_j^+ \underset{p_0}{\cup} e_0 \amalg \{\, \widetilde{P}(b_i, b_4^{-1}; b_j, b_k) \,\}, \qquad \widetilde{\mathrm{Per}}{}_j^{e}(\theta; n) = \emptyset \qquad (n > 1), \tag{55}$$

where $\widetilde{\ell}_j^+ \underset{p_0}{\cup} e_0$ indicates that the curves $\widetilde{\ell}_j^+$ and e_0 meet in the point p_0.

Next we consider $\widetilde{\mathrm{Fix}}_i(\theta)$ and $\widetilde{\mathrm{Per}}{}_i^{e}(\theta; n)$ on the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_3)_i$. If we take a parametrization of e_0 such that $t = 0$ and $t = \infty$ correspond to the points p_j and p_k respectively, then a simple check shows that the transformation $\tilde{g}_i^2$ on e_j is expressed as $t \mapsto b_4^{-2} t$. There is a parametrization of e_j such that $t = 0$ corresponds to p_j and $\tilde{g}_i^2$ is given by $t \mapsto b_4^2 t$. Since $b_4^2 \neq 1$, the transformation $\tilde{g}_i^2$ has exactly two fixed points on e_j, one of which is just p_j and the other is denoted by p_{ij}. Similarly, there is a parametrization of e_k such that $t = 0$ corresponds to p_k and $\tilde{g}_i^2$ is given by $t \mapsto b_4^{-2} t$, and hence $\tilde{g}_i^2$ has exactly two fixed points on e_k, one of which is just p_k and the other is denoted by p_{ik}. There are no smooth fixed points $\widetilde{\mathrm{Fix}}{}_i^{\circ}(\theta)$, because the smoothness condition of Table 3 with (i, j, k) replaced by (k, i, j) is not satisfied for any labels there. So we have $\widetilde{\mathrm{Fix}}_i(\theta) = \widetilde{\mathrm{Fix}}{}_i^{e}(\theta) = \{p_j,\, p_{ij},\, p_k,\, p_{ik}\}$ and $\widetilde{\mathrm{Fix}}{}_i^{\circ}(\theta) = \emptyset$ on the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_3)_i$. By symmetry there is a similar characterization of $\widetilde{\mathrm{Fix}}_k(\theta)$ on the same stratum. By permuting the indices (i, j, k), we have

$$\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\mathrm{Fix}}{}_j^{e}(\theta) = \{\text{four points}\}, \qquad \widetilde{\mathrm{Fix}}{}_j^{\circ}(\theta) = \emptyset, \tag{56}$$

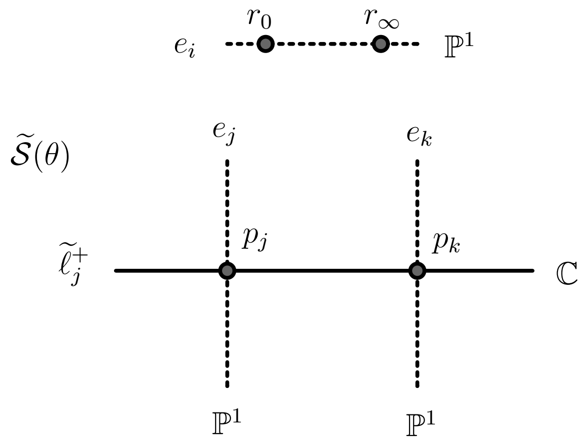


Figure 15: Surface of type $A_1^{\oplus 3}$

on the $\widetilde{W}(D_4^{(1)})$-strata of types $(A_3)_j$ and $(A_3)_k$. A slightly further consideration yields

$$\widetilde{\mathrm{Per}}^{\,e}_j(\theta; n) = \begin{cases} e_k \underset{p_k}{\cup} e_0 \underset{p_i}{\cup} e_i & \text{(if } b_4 \text{ is a primitive } 2n\text{-th root of unity)}, \\ \emptyset & \text{(otherwise)}, \end{cases}$$

on the stratum $(A_3)_j$ and a similar characterization of it on the stratum $(A_3)_k$.

Example 7.6 ($\boldsymbol{A_1^{\oplus 3}}$) Consider the $\widetilde{W}(D_4^{(1)})$-stratum of type $A_1^{\oplus 3}$. We may assume that $\kappa_i = \kappa_j = \kappa_k = 0$ so that $b_i = b_j = b_k = 1$. But we have $b_4 \not\in \{\pm 1\}$ since our stratum is not of type D_4 nor of type $A_1^{\oplus 4}$. In this case the surface $\mathcal{S}(\theta)$ has three singular points of type A_1 at $(x_i, x_j, x_k) = (b_4 + b_4^{-1}, 2, 2)$, $(2, b_4 + b_4^{-1}, 2)$, $(2, 2, b_4 + b_4^{-1})$, which are called q_i, q_j, q_k respectively. Note that the two points q_j and q_k lie on the line ℓ_j^+ but q_i does not lie on the union $\ell_j^+ \amalg \ell_j^-$. The minimal resolution (15) is obtained by blowing up these three points (see Figure 15). First, consider the blowing-up at q_k and represent it by $(x_i, x_j, x_k) = (u_i u_j + 2, u_j + 2, u_k u_j + b_4 + b_4^{-1})$. Then the strict transform $\widetilde{\ell}_j^+$ of the line ℓ_j^+ is given by $u_i = u_k + 1 = 0$, while the exceptional curve e_k is given by $b_4(u_i + u_k + 1)^2 + (b_4 - 1)^2 u_i = 0$. The curves $\widetilde{\ell}_j^+$ and e_k intersect in the point $(u_i, u_j, u_k) = (0, 0, -1)$; this point is called p_k. If we parametrize the curve e_k as

$$u_i = -\frac{b_4}{(b_4 - 1)^2 t^2}, \quad u_j = 0, \quad u_k = -\frac{\{(b_4 - 1)t + 1\}\{(b_4 - 1)t - b_4\}}{(b_4 - 1)^2 t^2} \qquad (t \in \mathbb{P}^1),$$

where $t = \infty$ corresponds to the point p_k, then the lifted transformation $\widetilde{g}_j^2$ induces the shift $t \mapsto t + 1$ and hence acts on e_k as a parabolic Möbius transformation fixing p_k only. In a similar manner $\widetilde{g}_j^2$ acts on the exceptional curve e_j over q_j as a parabolic Möbius transformation fixing only the intersection point p_j of $\widetilde{\ell}_j^+$ and e_j. Next we consider the blowing-up at q_i and represent it by $(x_i, x_j, x_k) = (u_i u_j + b_4 + b_4^{-1}, u_j + 2, u_k u_j + 2)$. Then the exceptional curve e_i is given by $b_4(u_i + u_k + 1)^2 + (b_4 - 1)^2 u_k = 0$, which can be parametrized as

$$u_i = -\frac{(b_4 + 1)^2 t}{(b_4 t + 1)^2}, \quad u_j = 0, \quad u_k = -\frac{b_4 (t - 1)^2}{(b_4 t + 1)^2} \qquad (t \in \mathbb{P}^1).$$

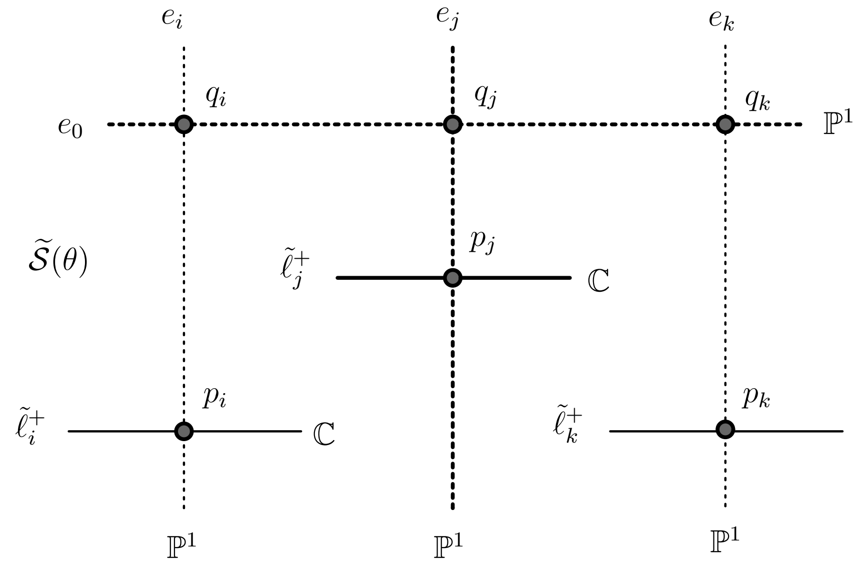


Figure 16: Surface of type D_4

In terms of this parametrization, the transformation $\widetilde{g}_j^2$ restricts to the map $t \mapsto b_4^2 t$ on the exceptional curve e_i. Let r_0 and r_∞ be the points on e_i corresponding to $t = 0$ and $t = \infty$ respectively. Since $b_4^2 \neq 1$, the map $\widetilde{g}_j^2$ acts on e_i as a Möbius transformation with exactly two fixed points r_0 and r_∞. Hence the set $\widetilde{\mathrm{Fix}}_j(\theta)$ contains the line component $\tilde{\ell}_j^+$ and the Riccati component $\{r_0, r_\infty\}$, but has no smooth-point component, since $F(b_i, b_4; b_j, b_k) = F(b_i, b_4^{-1}; b_j, b_k) = b_4 + b_4^{-1} \not\in \{\pm 2\}$ is a double root of the quartic equation (39) (see Theorem 6.3). Thus we have

$$\widetilde{\mathrm{Fix}}_j(\theta) = \tilde{\ell}_j^+ \amalg \{r_0, r_\infty\}. \tag{57}$$

As for the Riccati periodic points, since $\widetilde{g}_j^2$ acts on e_i as $t \mapsto b_4^2 t$, we have for any $n > 0$,

$$\widetilde{\mathrm{Per}}_j^{e}(\theta; n) = \begin{cases} e_i & \text{(if } b_4 \text{ is a primitive } 2n\text{-th root of unity)}, \\ \emptyset & \text{(otherwise)}. \end{cases}$$

Example 7.7 (D_4) Consider the $\widetilde{W}(D_4^{(1)})$-stratum of type D_4, say, the $W(D_4^{(1)})$-stratum with value $\theta = (8, 8, 8, 28)$. In this case the surface $\mathcal{S}(\theta)$ has only one singular point of type D_4 at $(x_i, x_j, x_k) = (2, 2, 2)$. The minimal resolution (15) is obtained by successive blowing-ups: Blow up the singular point. If we express the blowing-up as $(x_i, x_j, x_k) = (u_i u_j + 2, u_j + 2, u_k u_j + 2)$ in terms of coordinates (u_i, u_j, u_k), then the strict transform of the surface $\mathcal{S}(\theta)$ is represented as $u_i u_j u_k + (u_i + u_k + 1)^2 = 0$. The exceptional curve e is given by $u_j = u_i + u_k + 1 = 0$. The strict transforms of ℓ_i^+ and ℓ_j^+ are given by $u_i + 1 = u_k = 0$ and $u_i = u_k + 1 = 0$, while the strict transform of ℓ_k^+ is at infinity and not expressible in terms of the coordinates (u_i, u_j, u_k). The blow-up surface has three singularities, all of which are of type A_1 and located at the points in which the exceptional curve e intersects the strict transforms of ℓ_i^+, ℓ_j^+, ℓ_k^+. The lifts of the transformations g_i^2, g_j^2, g_k^2 fix the curve e pointwise, since they fix the three singular points on it. Again blow up these points. Then we obtain a minimal resolution (15) of the surface $\mathcal{S}(\theta)$ as depicted in Figure 16, where e_i, e_j, e_k are the exceptional curves over the singular points and

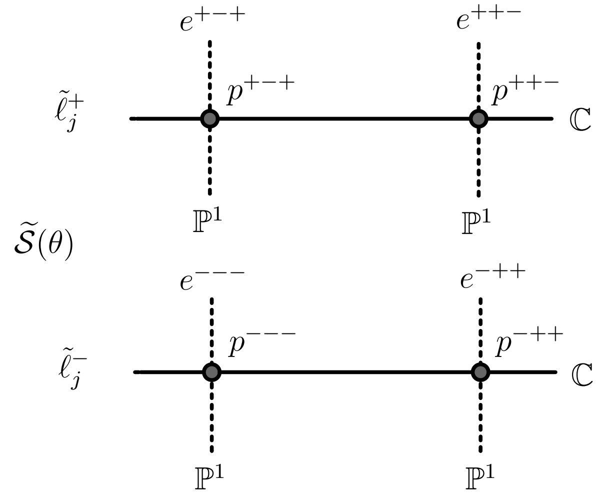


Figure 17: Surface of type $A_1^{\oplus 4}$

e_0, $\widetilde{\ell}_i^+$, $\widetilde{\ell}_j^+$, $\widetilde{\ell}_k^+$ are the strict transforms of e, ℓ_i^+, ℓ_j^+, ℓ_k^+, respectively. Being the strict transform of e, the exceptional curve e_0 is fixed pointwise by the lifts $\tilde{g}_i^2$, $\tilde{g}_j^2$, $\tilde{g}_k^2$ of g_i^2, g_j^2, g_k^2, and hence carries rational solutions. Moreover the lift $\tilde{g}_j^2$ fixes e_j pointwise. This can be seen without computation. Since $\tilde{g}_j^2$ is area-preserving and fixes $\widetilde{\ell}_j^+$ pointwise, it has derivative 1 at p_j along the curve e_j. So the Möbius transformation on e_j induced by $\tilde{g}_j^2$ is either identity or a map of parabolic type. But the latter is impossible because it has at least two fixed points at p_j and q_j (see Figure 16). Hence $\widetilde{g}_j^2$ acts on e_j as the identity. Next we shall observe that $\widetilde{g}_j^2$ acts on e_i as a parabolic Möbius transformation fixing q_i only. If we express the blowing-up at $(u_i, u_j, u_k) = (-1, 0, 0)$ as $(u_i, u_j, u_k) = (v_i v_k - 1, v_j v_k, v_k)$, then the exceptional curve e_i is given by $v_k = v_j - (v_i + 1)^2 = 0$. Parametrize e_i as $(v_i, v_j, v_k) = (-(t+1)/t, t^{-2}, 0)$, where q_i corresponds to $t = \infty$. Then $\widetilde{g}_j^2$ acts on e_i by the shift $t \mapsto t + 1$. Similarly $\widetilde{g}_j^2$ acts on e_k as a parabolic transformation fixing q_k only. By symmetry, $\widetilde{g}_i^2$ and $\widetilde{g}_k^2$ act on e_j as parabolic transformations fixing q_j only. Notice that the exceptional curve e_0 carries rational Riccati solutions, while $e_j - \{q_j\}$ carries Riccati solutions of infinte period. Thus we have

$$\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\ell}_j^+ \underset{p_j}{\cup} e_j \underset{q_j}{\cup} e_0, \qquad \widetilde{\mathrm{Per}}_j^{\,e}(\theta; n) = \emptyset \quad (n > 1). \tag{58}$$

Example 7.8 ($A_1^{\oplus 4}$) Consider the $\widetilde{W}(D_4^{(1)})$-stratum of type $A_1^{\oplus 4}$, where $\theta = (0, 0, 0, -4)$ and $\mathrm{Fix}_j(\theta) = \ell_j^+ \amalg \ell_j^-$. In this case the surface $\mathcal{S}(\theta)$ has four singularities of type A_1 at $(x_i, x_j, x_k) = (2\varepsilon_i, 2\varepsilon_j, 2\varepsilon_k) \in \{\pm 2\}^3$ with $\varepsilon_i \varepsilon_j \varepsilon_k = -1$. Blow up at these points to obtain a minimal resolution as in (15). Let $e^{\varepsilon_i \varepsilon_j \varepsilon_k}$ be the exceptional line over $(x_i, x_j, x_k) = (2\varepsilon_i, 2\varepsilon_j, 2\varepsilon_k)$ and $\widetilde{\ell}_j^{\varepsilon_i}$ be the strict transform of $\ell_j^{\varepsilon_i}$. Moreover let $p^{\varepsilon_i \varepsilon_j \varepsilon_k}$ denote the intersection point of the lines $e^{\varepsilon_i \varepsilon_j \varepsilon_k}$ and $\widetilde{\ell}_j^{\varepsilon_i}$ (see Figure 17). Then the lifted transformation $\tilde{g}_j^2 : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ acts on the exceptional line $e^{\varepsilon_i \varepsilon_j \varepsilon_k} \cong \mathbb{P}^1$ as a Möbius transformation. It is a parabolic transformation with the only fixed point $p^{\varepsilon_i \varepsilon_j \varepsilon_k}$. Let us check this for $(\varepsilon_i, \varepsilon_j, \varepsilon_k) = (-1, -1, -1)$. The blowing-up of $\mathbb{C}^3$ at $(x_i, x_j, x_k) = (-2, -2, -2)$ is described by $x_i = u_i u_j - 2$, $x_j = u_j - 2$, $x_k = u_j u_k - 2$, in terms of coordinates (u_i, u_j, u_k) around $(0, 0, 0)$. Then the exceptional line e^{---} is represented by the

equations $u_j = 0$ and $(u_i - u_k)^2 - 2(u_i + u_k) + 1 = 0$ and hence it is parametrized as

$$u_i = \left(\frac{2}{t+1}\right)^2, \qquad u_j = 0, \qquad u_k = \left(\frac{t-1}{t+1}\right)^2,$$

where the fixed point p^{---} corresponds to $t = \infty$. Then we can check that $\tilde{g}_j^2$ acts on the line e^{---} as the translation $t \mapsto t+4$, as desired. Thus the only fixed points of $\tilde{g}_j^2$ on the exceptional set $\mathcal{E}(\theta)$ are the four points $p^{\varepsilon_i \varepsilon_j \varepsilon_k}$ with $\varepsilon_i \varepsilon_j \varepsilon_k = -1$ and there are no periodic points, so that

$$\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\ell}_j^+ \amalg \widetilde{\ell}_j^-, \qquad \widetilde{\mathrm{Per}}_j^e(\theta; n) = \emptyset \quad (n > 1). \tag{59}$$

8 Power Geometry

We apply the method of power geometry [5, 6, 7] to construct as many algebraic branch solutions to $\mathrm{P_{VI}}(\kappa)$ as possible around each fixed singular point. Basically we can follow the arguments of [7]. However, while the attention of [7] is restricted to generic parameters, we require a thorough treatment of all parameters, where much ampler varieties of patterns are present. Moreover, the way in [7] of representing the parameters of Painlevé VI is not convenient for our purpose. So we have to redevelop the necessary arguments on power geometry from scratch.

In view of Remark 5.1, it is sufficient to work around the origin $z = 0$. In order to apply the method in [5, 6, 7], we reduce the system (1) into a single second-order equation. If $(q, p) = (q(z), p(z))$ is a solution to system (1) such that $q \not\equiv 0,\ 1,\ z,\ \infty$, then we solve the first equation of system (1) with respect to $p = p(z)$ to obtain

$$p = \frac{z(z-1)q' + \kappa_1 q_1 q_z + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_z}{2 q_0 q_1 q_z}. \tag{60}$$

Substituting this into the second equation yields the single second-order equation

$$\begin{aligned} \frac{d^2 q}{dz^2} \;=\; & \frac{1}{2}\left(\frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_z}\right)\left(\frac{dq}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q_z}\right)\left(\frac{dq}{dz}\right) \\ & + \frac{q_0 q_1 q_z}{2z^2(z-1)^2}\left\{\kappa_4^2 - \kappa_1^2 \frac{z}{q_0^2} + \kappa_3^2 \frac{z-1}{q_1^2} + (1 - \kappa_2^2)\frac{z(z-1)}{q_z^2}\right\}. \end{aligned} \tag{61}$$

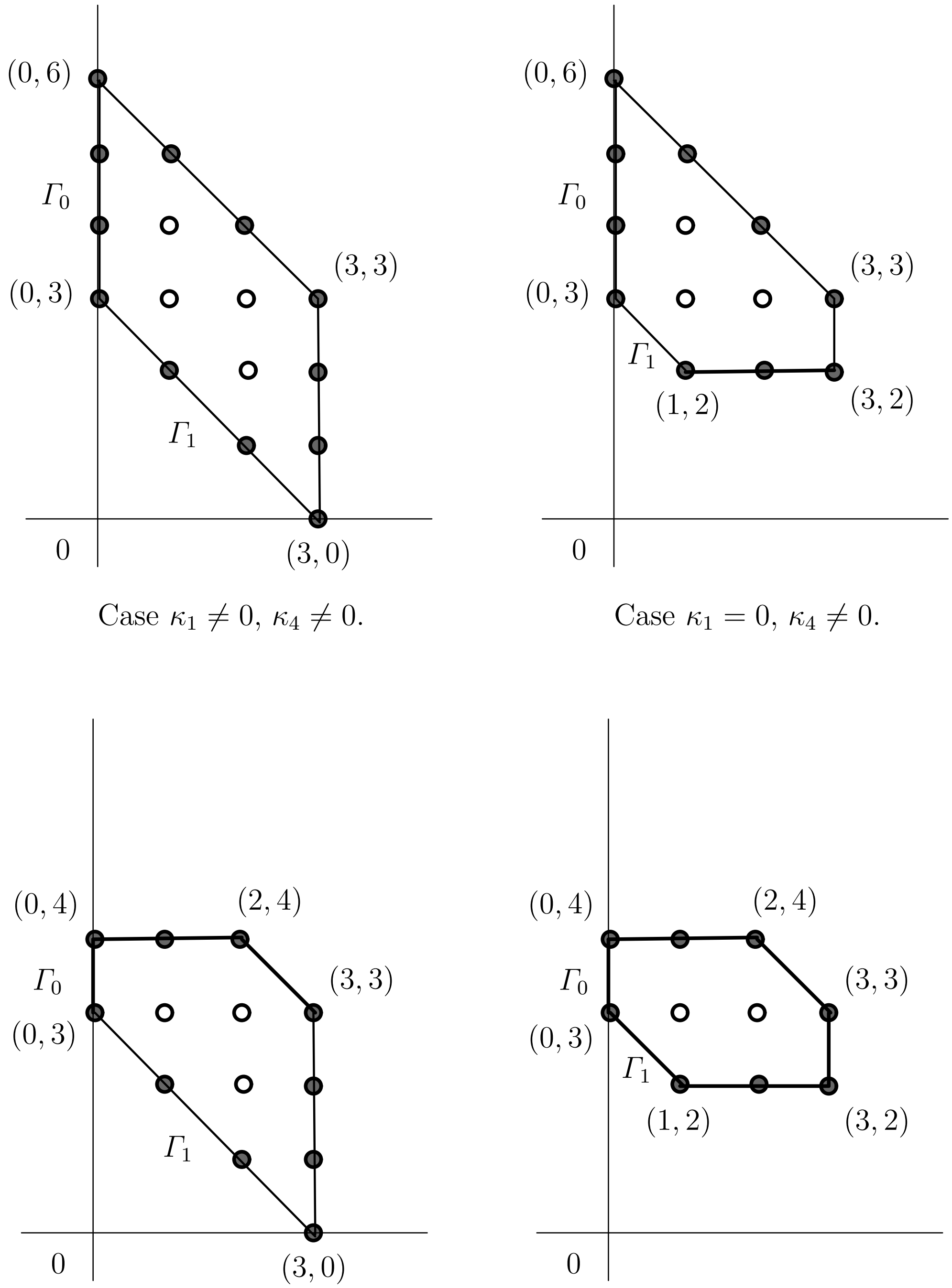
Multiply equation (61) by $2z^2(z-1)^2 q_0 q_1 q_z$ and move its right-hand side to the left to obtain

$$P(z, q) = 0, \tag{62}$$

where $P(z, q)$ is a polynomial of (z, q, q', q''), that is, a differential sum of (z, q), whose explicit formula is omitted here but can be found in [7]. Therefore system (1) is equivalent to equation (62) together with (60) except for the possible solutions such that $q \equiv 0,\ 1,\ z,\ \infty$. A simple check shows that the Newton polygon of equation (62) is given as in Figure 18, where there are four patterns according as the parameters κ_1 and κ_4 are zero or not.

First we search for a *holomorphic* solution germ $q = q(z)$ to equation (62) around $z = 0$. We have only to construct formal power series solutions of the form

$$q = c z^r + (\text{higher order terms}), \qquad (r, c) \in \mathbb{Z} \times \mathbb{C}^{\times}, \tag{63}$$



Case $\kappa_1 \neq 0$, $\kappa_4 \neq 0$.

Case $\kappa_1 = 0$, $\kappa_4 \neq 0$.

Case $\kappa_1 \neq 0$, $\kappa_4 = 0$.

Case $\kappa_1 = 0$, $\kappa_4 = 0$.

Figure 18: Newton polygon for Painlevé VI

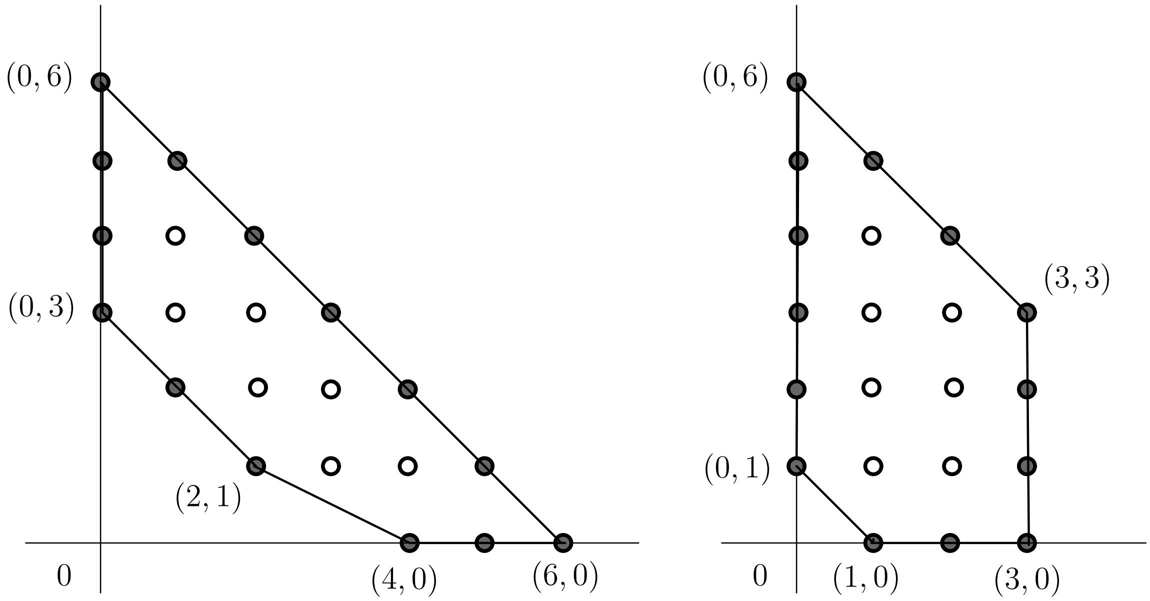


Figure 19: Newton polygons for Lemma 8.1 (left) and Lemma 8.2 (right)

since any formal power series solution to equation (62) is convergent [10, 11]. Then it follows from (60) that the associated formal Laurent series for $p = p(z)$ is also convergent.

In order to construct formal solutions (63), we consider the truncations along the edges Γ_1 and Γ_0 of the Newton polygons in Figure 18. We see that Γ_1 and Γ_0 have outer normal vectors $(p_1, p_2) = (-1, -1)$ and $(p_1, p_2) = (-1, 0)$, whose slopes are $p_2/p_1 = 1$ and $p_2/p_1 = 0$ respectively. Thus the edges Γ_1 and Γ_0 correspond to the exponents $r = 1$ and $r = 0$ respectively. The truncation of $P = P(z, q)$ along the edge Γ_1 is given by

$$\begin{aligned} P_1 &= -2zq^2q' + 2z^2q(q')^2 - 2z^2q^2q'' + (\kappa_1^2 - \kappa_2^2 + 1)zq^2 - z^3(q')^2 + 2z^3qq'' \\ &\quad -2\kappa_1^2z^2q + \kappa_1^2z^3. \end{aligned}$$

Substituting $q = cz$ into equation $P_1 = 0$ yields $c = \kappa_1/(\kappa_1 + \varepsilon\kappa_2)$ with any sign $\varepsilon \in \{\pm 1\}$. Similarly, the truncation of $P = P(z, q)$ along the edge Γ_0 is given by

$$\begin{aligned} P_0 &= -2zq^2q' + 2z^2q(q')^2 - 2z^2q^2q'' + (\kappa_3^2 - \kappa_4^2)q^4 + 2zq^3q' - 3z^2q^2(q')^2 \\ &\quad +2z^2q^3q'' + 2\kappa_4^2q^5 - \kappa_4^2q^6. \end{aligned}$$

Substituting $q = c$ into equation $P_0 = 0$ yields $c = (\kappa_4 + \varepsilon\kappa_3)/\kappa_4$ with any sign $\varepsilon \in \{\pm 1\}$.

Lemma 8.1 *If $\kappa_1 + \kappa_2 \notin \mathbb{Z}$, then there exists a holomorphic solution around the origin $z = 0$,*

$$q = \frac{\kappa_1 z}{\kappa_1 + \kappa_2} + \kappa_1\kappa_2 \sum_{k=2}^{\infty} a_{k,+}(\kappa)\, z^k, \qquad p = \kappa_0(\kappa_0 + \kappa_4) \sum_{k=0}^{\infty} b_{k,+}(\kappa)\, z^k, \tag{64}$$

depending holomorphically on $\kappa \in \mathcal{K}$ with $\kappa_1 + \kappa_2 \notin \mathbb{Z}$. Similarly, if $\kappa_1 - \kappa_2 \notin \mathbb{Z}$, then there exists a meromorphic solution around the origin $z = 0$,

$$q = \frac{\kappa_1 z}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2 \sum_{k=2}^{\infty} a_{k,-}(\kappa)\, z^k, \qquad p = \frac{\kappa_1 - \kappa_2}{z} + \sum_{k=0}^{\infty} b_{k,-}(\kappa)\, z^k, \tag{65}$$

depending holomorphically on $\kappa \in \mathcal{K}$ with $\kappa_1 - \kappa_2 \notin \mathbb{Z}$.

Proof. Substituting $q = \kappa_1 z(\kappa_1 + \varepsilon\kappa_2)^{-1} + \kappa_1\kappa_2 Q$ with $\varepsilon \in \{\pm 1\}$ into equation (62) yields

$$(\kappa_1 + \varepsilon\kappa_2)^6 P\left(z, \kappa_1 z(\kappa_1 + \varepsilon\kappa_2)^{-1} + \kappa_1\kappa_2 Q\right) = \kappa_1^2\kappa_2^2\, p(z, Q),$$

where $p(z; Q)$ is a differential sum of (z, Q) with coefficients in $\mathbb{C}[\kappa]$ whose Newton polygon is given as in Figure 19 (left). The vertex $(2, 1)$ carries the linear differential expression

$$\mathcal{L}_\varepsilon Q = 2\varepsilon(\kappa_1 + \varepsilon\kappa_2)^4 x^2\{x^2 Q'' - xQ' - (\kappa_1 + \varepsilon\kappa_2 + 1)(\kappa_1 + \varepsilon\kappa_2 - 1)Q\},$$

while the vertex $(4, 0)$ carries the monomial $(\kappa_1 + \varepsilon\kappa_2)^2\{(\kappa_1 + \varepsilon\kappa_2)^2 + \kappa_3^2 - \kappa_4^2 - 1\}x^4$. The corresponding characteristic polynomial is given by

$$v_\varepsilon(k) = 2\varepsilon(\kappa_1 + \varepsilon\kappa_2)^4(k - 1 - \kappa_1 - \varepsilon\kappa_2)(k - 1 + \kappa_1 + \varepsilon\kappa_2).$$

Hence $1 + |\kappa_1 + \varepsilon\kappa_2|$ is the unique critical value of the problem. If it is not an integer, then the coefficients $a_{k,\varepsilon}(\kappa)$ of the expansions (64) and (65) are determined uniquely and recursively. By substituting the resulting power series $q = q(z)$ into equation (60), the Laurent series for $p = p(z)$ is uniquely determined as in (64) and (65). As is mentioned earlier, the formal solutions (64) and (65) so obtained are convergent. □

Lemma 8.2 *Assume that κ_4 is nonzero. If $\kappa_4 + \kappa_3 \notin \mathbb{Z}$, then there exists a holomorphic solution germ around the origin $z = 0$,*

$$q = \frac{\kappa_4 + \kappa_3}{\kappa_4} + \frac{\kappa_3}{\kappa_4}\sum_{k=1}^{\infty} a_{k,+}(\kappa)\, z^k, \qquad p = -\kappa_4\kappa_0 \sum_{k=0}^{\infty} b_{k,+}(\kappa)\, z^k, \tag{66}$$

depending holomorphically on $\kappa \in \mathcal{K}$ with $\kappa_4 \neq 0$ and $\kappa_4 + \kappa_3 \notin \mathbb{Z}$. Similarly, if $\kappa_4 - \kappa_3 \notin \mathbb{Z}$ then there exists a holomorphic solution germ around the origin $z = 0$,

$$q = \frac{\kappa_4 - \kappa_3}{\kappa_4} + \frac{\kappa_3}{\kappa_4}\sum_{k=1}^{\infty} a_{k,-}(\kappa)\, z^k, \qquad p = -\kappa_4(\kappa_0 + \kappa_4) \sum_{k=0}^{\infty} b_{k,-}(\kappa)\, z^k, \tag{67}$$

depending holomorphically on $\kappa \in \mathcal{K}$ with $\kappa_4 \neq 0$ and $\kappa_4 - \kappa_3 \notin \mathbb{Z}$.

Proof. Substituting $q = \kappa_4^{-1}(\kappa_4 + \varepsilon\kappa_3) + \kappa_4^{-1}\kappa_3 Q$ into equation (62) yields

$$\kappa_4^4 P\left(z, \kappa_4^{-1}(\kappa_4 + \varepsilon\kappa_3) + \kappa_4^{-1}\kappa_3 Q\right) = \kappa_3^2\, p(z; Q),$$

where $p(z; Q)$ is a differential sum of (z, Q) with coefficients in $\mathbb{C}[\kappa]$. The vertex $(0, 1)$ carries the linear differential expression $\mathcal{L}_\varepsilon Q = 2(\kappa_4 + \varepsilon\kappa_3)^2\{x^2 Q'' + xQ' - (\kappa_4 + \varepsilon\kappa_3)^2 Q\}$, while the vertex $(1, 0)$ carries the monomial $(\kappa_4 + \varepsilon\kappa_3)^2\{1 + \kappa_1^2 - \kappa_2^2 - (\kappa_4 + \varepsilon\kappa_3)^2\}x$. The corresponding characteristic polynomial is given by $v_\varepsilon(k) = 2(\kappa_4 + \varepsilon\kappa_3)^2\{k - (\kappa_4 + \varepsilon\kappa_3)\}\{k + (\kappa_4 + \varepsilon\kappa_3)\}$. Hence $|\kappa_4 + \varepsilon\kappa_3|$ is the unique critical value of the problem. If it is not an integer, then the coefficients $a_{k,\varepsilon}(\kappa)$ of expansions (66) and (67) are determined uniquely and recursively. Then substituting the resulting series for $q = q(z)$ into equation (60) yields the Laurent series for $p = p(z)$ as in (66) and (67). The formal solutions so obtained are convergent. □

The solutions in Lemmas 8.1 and 8.2 are essentially constructed in [7, 23]. We construct more particular solutions for the parameters on various strata of higher codimensions.

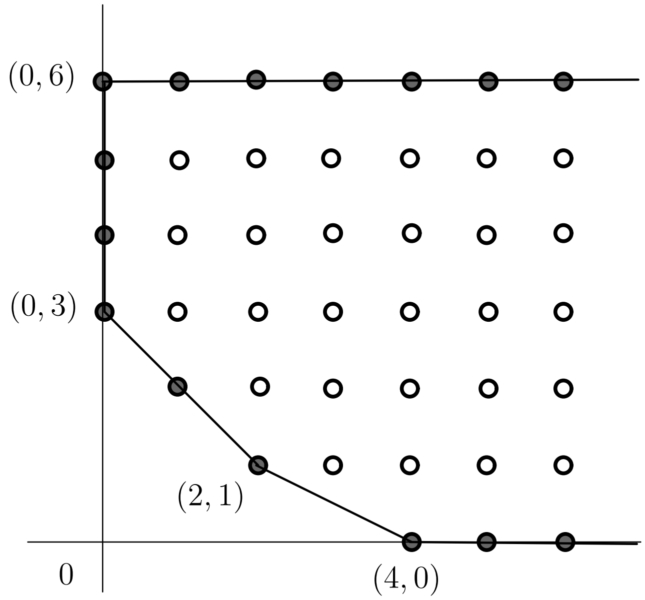


Figure 20: Newton polygon for Lemma 8.3

Lemma 8.3 ($A_1^{\oplus 2}$ and $A_1^{\oplus 3}$) *If $\kappa_1 = \kappa_2 = 0$, then there exists a 1-parameter family of holomorphic solution around the origin $z = 0$,*

$$\begin{aligned} q &= \frac{tz}{t + (1-t)(1-z)^{\kappa_4}} + t(1-t)\kappa_0(\kappa_0+\kappa_3)\sum_{k=2}^{\infty} a_k(t;\kappa)\, z^k, \\ p &= \kappa_0(\kappa_0+\kappa_4)\sum_{k=0}^{\infty} b_k(t;\kappa)\, z^k, \end{aligned} \tag{68}$$

depending on $t \in \mathbb{C}$, where $a_2(t;\kappa) = 2$, $b_0(t;\kappa) = 1$ and the remaining coefficients $a_k(t;\kappa)$, $k \geq 3$, and $b_k(t;\kappa)$, $k \geq 1$, are polynomials of (t, κ_3, κ_4) determined uniquely and recursively.

Proof. We put $R(z;t) = tz\{t+(1-t)(1-z)^{\kappa_4}\}^{-1}$. Substituting $q = R(z;t)+t(1-t)\kappa_0(\kappa_0+\kappa_3)Q$ into equation (62) and multiplying the result by $\{t + (1-t)(1-z)^{\kappa_4}\}^4$ yield

$$\begin{aligned} p(z, Q; t) &:= \{t + (1-t)(1-z)^{\kappa_4}\}^4 P(z, R(z;t) + t(1-t)\kappa_0(\kappa_0+\kappa_3)Q) \\ &= 2t^2(1-t)^2\kappa_0(\kappa_0+\kappa_3)\{\mathcal{L}Q + g(z, Q; t) + h(z)\} = 0, \end{aligned} \tag{69}$$

where $\mathcal{L}Q = z^2\{z^2Q'' - zQ' + Q\}$ and $h(z) = -2z^4(1-z)^{2\kappa_4+1}$. The Newton polygon of (69) is given as in Figure 20, where the terms $\mathcal{L}Q$ and $h(z)$ correspond to the vertex $(2,1)$ and the horizontal infinite edge emanating from the vertex $(4,0)$ respectively, and the remaining term $g(z, Q; t)$ corresponds to the remaining part of the polygon. Since the characteristic equation of $\mathcal{L}Q$ is $(k-1)^2 = 0$ having the unique root $k = 1$, the coefficients $a_k(t;\kappa)$, $k \geq 2$, in (68) are determined uniquely and recursively. Here the leading coefficient $a_2(t;\kappa)$ is found to be $a_2(t;\kappa) = 2$ by substituting $Q = a_2(t;\kappa)z^2$ into the truncation $\mathcal{L}Q - 2z^4 = 0$ of equation (69) along the edge connecting the vertices $(2,1)$ and $(4,0)$. Substituting the resulting series $q = q(z)$ into (60) we have $p = p(z)$ as in (68). $\square$

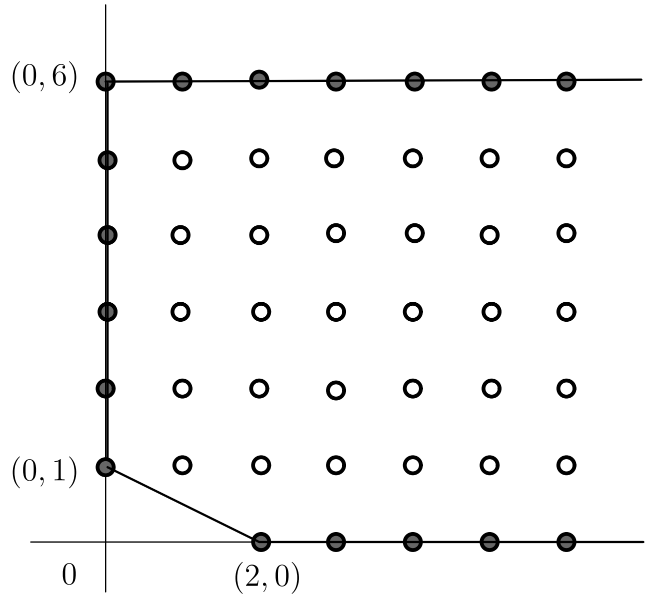


Figure 21: Newton polygon for Lemma 8.4

Lemma 8.4 ($\boldsymbol{A_3}$) *If $\kappa_0 = \kappa_1 = \kappa_2 = 0$, $\kappa_3 + \kappa_4 = 1$, then there exists a 1-parameter family of holomorphic solutions around the origin $z = 0$,*

$$q = \frac{z}{1-(1-z)^{\kappa_4}} + t\kappa_3 z + t\kappa_3\kappa_4 \sum_{k=2}^{\infty} a_k(t;\kappa)\, z^k, \qquad p = -t\kappa_4^3 \sum_{k=1}^{\infty} b_k(t;\kappa)\, z^k, \tag{70}$$

depending on $t \in \mathbb{C}$, where $a_2(t;\kappa) = t\kappa_3$, $b_1(t;\kappa) = 1$ and the coeffcients $a_k(t;\kappa)$, $k \geq 2$, and $b_k(t;\kappa)$, $k \geq 1$, are polynomials of (t, κ_4) determined uniquely and recursively.

Proof. Put $R(z) = z\{1-(1-z)^{\kappa_4}\}^{-1}$. Substituting $q = R(z) + t\kappa_3 z + t\kappa_3\kappa_4 Q$ into (62) yields

$$P(z, R(z) + t\kappa_3 z + t\kappa_3\kappa_4 Q) = t\kappa_3^2\kappa_4^2 R(x)^4\, p(z, Q; t),$$

where $p(z, Q; t)$ is a differential sum of (z, Q) with coefficients in $\mathbb{C}[t, \kappa_4]$ whose Newton polygon is given as in Figure 21. Especially the vertex $(0, 1)$ carries the linear differential expression $\mathcal{L}Q = 2(z^2 Q'' + zQ' - Q)$, whose characteristic polynomial is $2(k-1)(k+1)$, while the vertex $(2, 0)$ carries the monomial $-6t\kappa_3 z^2$. Since the critical values $k = \pm 1$ are smaller than 2, the coefficients $a_k(t;\kappa)$, $k \geq 2$, in (70) are determined uniquely and recursively, where the leading coefficient $a_2(t;\kappa)$ is found to be $t\kappa_3$. The rest of the proof is similar to that in Lemma 8.3. □

Lemma 8.5 ($\boldsymbol{D_4}$) *If $\kappa_0 = \kappa_1 = \kappa_2 = \kappa_3 = 0$ and $\kappa_4 = 1$, then there exists a 1-parameter family of holomorphic solution germs around the origin $z = 0$,*

$$q = 1 + t\sum_{k=1}^{\infty} a_k(t)\, z^k, \qquad p = \frac{z}{(1-z)\log(1-z)} + t\sum_{k=1}^{\infty} b_k(t)\, z^k, \tag{71}$$

depending on a parameter $t \in \mathbb{C}$, where $a_1(t) = b_1(t) = 1$ and the remaining coeffieicents $a_k(t)$ and $b_k(t)$, $k \geq 2$, are polynomials of t determined uniquely and recursively.

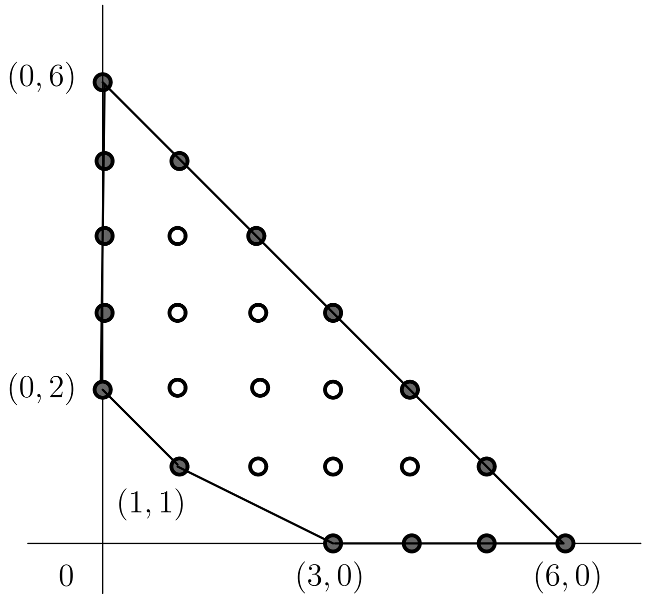


Figure 22: Newton polygons for Lemma 8.5 (left) and Lemma 8.6 (right)

Proof. Substituting $q = 1 + tz + tQ$ into equation (62) yields $P(z, 1 + tQ) = t^2 p(z, Q; t)$, where $p(z, Q; t)$ is a differential sum of (z, Q) with coefficients in $\mathbb{C}[t]$ whose Newton polygon is given as in Figure 22 (left). Consider the edge connecting $(1, 1)$ and $(3, 0)$. The vertex $(1, 1)$ carries the differential monomial $\mathcal{L}Q = 2z^3 Q''$, whose characteristic polynomial is $k(k-1)$, while the vertex $(3, 0)$ carries the monomial $-4tz^3$. Put $a_1(t) = 1$. Since the critical values $k = 0, 1$ are smaller than 2, the coefficients $a_k(t)$, $k \geq 2$, in (71) are determined uniquely and recursively. Substituting the resulting series $q = q(z)$ into (60) we have $p = p(z)$ as in (71). Here the term $z/\{(1-z)\log(1-z)\}$ is singled out from $p = p(z)$, because putting $t = 0$ yields the special solution $q \equiv 1$ and $p = z/\{(1-z)\log(1-z)\}$ (see also Lemma 9.4). $\square$

Lemma 8.6 ($\boldsymbol{A_1^{\oplus 4}}$) *Let $\kappa_0 = 1/2$ and $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 0$. Then there exists a 1-parameter family of holomorphic solutions around the origin $z = 0$ depending on a parameter $t \in \mathbb{C}$,*

$$q = tz + t(1-t)\sum_{k=2}^{\infty} a_k(t)\, z^k, \qquad p = \sum_{k=0}^{\infty} b_k(t)\, z^k, \tag{72}$$

where the coefficients $a_k(t)$ and $b_k(t)$ are polynomials of t beginning with $a_2(t) = 1/2$ and $b_0(t) = 1/4$. Moreover there is another 1-parameter family of solutions around $z = 0$,

$$q = \frac{1}{t} + \frac{t-1}{t}\sum_{k=1}^{\infty} c_k(t)\, z^k, \qquad p = t\sum_{k=0}^{\infty} d_k(t)\, z^k, \tag{73}$$

depending on $t \in \mathbb{C}$, where $c_k(t)$ and $d_k(t)$ are polynomials of t beginning with $c_1(t) = 1/2$ and $d_0(t) = -1/2$. For $t = 0$, formula (73) represents the solution such that $q \equiv \infty$ and $p \equiv 0$.

Proof. We only derive (73), as (72) is derived in a similar manner. Substituting $q = t^{-1} + t^{-1}(t-1)Q$ into (62) yields $P(z, t^{-1} + t^{-1}(t-1)Q) = t^{-4}(t-1)^2 z^{-1} p(z, Q; t)$, where $p(z, Q; t)$ is a differential sum of (z, Q) with coefficients in $\mathbb{C}[t]$ whose Newton polygon is given as in Figure 22

(right). Consider the edge connecting $(0,1)$ and $(1,0)$. The vertex $(0,1)$ carries the differential sum $\mathcal{L}Q = -2(zQ' + z^2Q'')$, whose characteristic polynomial is $-2k^2$, while the vertex $(1,0)$ carries the monomial z. Since the critical value $k=0$ is smaller than 1, the coefficients $c_k(t)$, $k \ge 1$, in (73) are determined uniquely and recursively, where the leading term is $c_1(t) = 1/2$. Substituting the resulting series $q = q(z)$ into (60) we have $p = p(z)$ as in (73). $\square$

So far we have considered the edges $\varGamma_1$ and $\varGamma_0$ of the Newton polygon in Figure 18 and constructed meromorphic solutions around the origin $z=0$. Now let us consider the vertex $(0,3)$ of the polygon, which gives rise to algebraic branch solutions around $z=0$. The truncation of the differential sum $P = P(z,q)$ at the vertex $(0,3)$ is given by

$$P_3 = -2zq^2q' + 2z^2q(q')^2 - 2z^2q^2q''.$$

Its charactersistic polynomial $\chi(r)$ is defined by substituting $q = z^r$ into $P_3(z,q)$ and dividing the result by $q^3 = z^{3r}$. In the present situation we see that $\chi(r)$ is identically zero; $\chi(r) \equiv 0$. The normal cone of the vertex $(0,3)$ is $U_3 = \{\, (p_1,p_2) \in \mathbb{R}^2 \,:\, p_1 < 0,\, 0 < r = p_2/p_1 < 1\,\}$. Thus the truncated solutions at the vertex $(0,3)$ are $q = tz^r$ for an arbitrary $0 < r < 1$ and $t \in \mathbb{C}^\times$. The Fréchet derivative with respect to q at the truncated solution $q = tz^r$ is given by

$$\mathcal{L}_3Q = -2t^2z^{2r}\{z^2Q'' + (1-2r)zQ' + r^2Q\}.$$

The corresponding characteristic equation is given by $v_3(k) := -2t^2(k-r)^2 = 0$, which has the only root $k = r$. Let n be any integer greater than 1. In order to search for an algebraic n-branch solution around $z=0$, we take $r = m/n$ for any integer $0 < m < n$ coprime to n, and consider a formal Puiseux series solution of the form

$$q = tz^{m/n} + \sum_{\nu=m+1}^{\infty} a_\nu(t)\, z^{\nu/n}.$$

Since the characteristic equation $v_3(k) = 0$ has no roots such that $k > m/n$, the coefficients $a_\nu = a_\nu(t)$ can be determined uniquely and recursively for any given initial coefficient $t \in \mathbb{C}^\times$. The convergence of the formal solution and its holomorphic dependence on parameters follow easily if we rewrite the equation (62) in terms of the new independent variable $\zeta = z^{1/n}$ and apply the convergence arguments in [10, 11]. Thus we have established the following lemma.

Lemma 8.7 *For any integer $n > 1$, there exist $\varphi(n)$ mutually disjoint 1-parameter families of n-branch solution germs to $\mathrm{P_{VI}}(\kappa)$ around the origin $z=0$,*

$$\begin{aligned} q(z) &= tz^{m/n} + \sum_{\nu=m+1}^{\infty} a_\nu(n,m,t;\kappa)\, z^{\nu/n}, \\ p(z) &= \frac{m + n(\kappa_1+\kappa_2-1)}{2nt}\, z^{-m/n} + \sum_{\nu=-m+1}^{\infty} b_\nu(n,m,t;\kappa)\, z^{\nu/n}, \end{aligned} \tag{74}$$

where the discrete parameter m ranges over all integers $0 < m < n$ coprime to n and the continuous parameter t takes any value of the punctured complex line $\mathbb{C}^\times$.

Remark 8.8 The family (74) contains no Riccati solutions, even if $\kappa \in$ **Wall**. This will be shown in the proof of Lemma 9.14.

9 Injection Implies Surjection

We establish Theorem 1.3 based on the main idea described in §1. Due to the S_3-symmetry permuting the three fixed singular points, it suffices to work around $z = 0$ (see Remark 5.1).

As a preliminary we begin by constructing some Riccati solutions to equation (1). Assume that $\kappa_0 = 0$ so that $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$. Then the second equation of system (1) has the null solution $p(z) \equiv 0$. Substituting this into the first equation yields the Riccati equation

$$z(z-1)q' + \kappa_1 q_1 q_z + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_z = 0.$$

If κ_4 is nonzero, then the change of dependent variable

$$q = \frac{z(1-z)}{\kappa_4} \frac{d}{dz} \log\{(1-z)^{-\kappa_4} f\}$$

transfers the Riccati equation to the Gauss hypergeometric equation

$$z(1-z)f'' + \{(1-\kappa_3-\kappa_4) - (\kappa_2 - \kappa_4 + 1)z\}f' + \kappa_2\kappa_4 f = 0. \tag{75}$$

Next assume that $\kappa_0 = \kappa_1 = 0$ so that $\kappa_2 + \kappa_3 + \kappa_4 = 1$. In this case there is another type of Riccati solution to the system (1). The first equation of the system (1) has the null solution $q(z) \equiv 0$. Substituting this into the second equation yields the Riccati equation

$$z(z-1)p' + zp^2 + (\kappa_2 - 1 + \kappa_3 z)p = 0.$$

Then change of independent variable $p = (z-1)\dfrac{d}{dz}\log g$ takes it to the linear equation

$$z(1-z)g'' + \{(1-\kappa_2) - (\kappa_3 + 1)z\}g' = 0. \tag{76}$$

Lemma 9.1 (A_1) *Assume that $\kappa_0 = 0$, $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$, $\kappa_1 + \kappa_2 \not\in \mathbb{Z}$, $\kappa_3 + \kappa_4 \not\in \mathbb{Z}$ and $\kappa_1\kappa_4 \neq 0$. Then system (1) has two single-valued Riccati solutions around the origin $z = 0$,*

$$\begin{array}{lllll} \text{(a)} & q = \dfrac{\kappa_1 z}{\kappa_1 + \kappa_2} + O(\kappa_1 z^2), & & p \equiv 0, \\ \text{(b)} & q = \dfrac{\kappa_3 + \kappa_4}{\kappa_4} + O(\kappa_3 z), & & p \equiv 0. \end{array}$$

Proof. The solutions (a) and (b) are obtained from two linearly independent solutions

$$ {}_2F_1(\kappa_2, -\kappa_4, 1-\kappa_3-\kappa_4; z), \qquad z^{\kappa_3+\kappa_4} {}_2F_1(\kappa_2+\kappa_3+\kappa_4, \kappa_3, \kappa_3+\kappa_4+1; z),$$

of equation (75) repsectively. They also come from solutions (64) and (66) respectively. □

Lemma 9.2 (A_2) *Assume that $\kappa_0 = \kappa_1 = 0$, $\kappa_2 + \kappa_3 + \kappa_4 = 1$, $\kappa_2 \not\in \mathbb{Z}$, $\kappa_3 \neq 1$ and $\kappa_4 \neq 0$. Then system (1) has three single-valued Riccati solutions around the origin $z = 0$,*

$$\begin{array}{lllll} \text{(a)} & q \equiv 0, & & p \equiv 0, \\ \text{(b)} & q = \dfrac{\kappa_3 + \kappa_4}{\kappa_4} + O(z), & & p \equiv 0, \\ \text{(c)} & q \equiv 0, & & p = -\dfrac{\kappa_2(\kappa_3 - 1)}{\kappa_2 + 1} + O(z). \end{array}$$

Proof. The solutions (a) and (b) just come from (a) and (b) of Lemma 9.1 respectively, while solution (c) is obtained from the solution $z^{\kappa_2}{}_2F_1(\kappa_2, \kappa_2+\kappa_3, \kappa_2+1; z)$ of equation (76). $\square$

Lemma 9.3 (A_3) *Assume that $\kappa_0 = \kappa_1 = \kappa_2 = 0$, $\kappa_3 + \kappa_4 = 1$ and $\kappa_4 \neq 0$. Then system (1) admits a 1-parameter family of single-valued Riccati solutions around the origin $z = 0$,*

$$q(z; s) = \frac{s_0 z}{s_0 + s_1(1-z)^{\kappa_4}}, \quad p(z; s) \equiv 0, \qquad (s = [s_0 : s_1] \in \mathbb{P}^1). \tag{77}$$

Proof. The hypergeometric equation (75) becomes $(1-z)f'' - \kappa_3 f' = 0$, whose nontrivial solutions are given by $f(z) = s_0 + s_1(1-z)^{\kappa_4}$ with $(s_0, s_1) \in \mathbb{C}^2 - \{(0,0)\}$. The corresponding Riccati solutions are the 1-parameter family of single-valued solutions $q = q(z; s)$ as in (77). $\square$

Lemma 9.4 (D_4) *Assume that $\kappa_0 = \kappa_1 = \kappa_2 = \kappa_3 = 0$ and $\kappa_4 = 1$.*

(1) *System (1) admits a 1-parameter family of rational Riccati solutions*

$$q(z; s) = \frac{s_0 z}{s_0 + s_1(1-z)}, \quad p(z; s) \equiv 0, \qquad (s = [s_0 : s_1] \in \mathbb{P}^1). \tag{78}$$

(2) *System (1) admits a 1-parameter family of single-valued Riccati solutions around $z = 0$,*

$$q(z; t) \equiv 1, \quad p(z; t) = \frac{t_0 z}{(1-z)\{t_0 \log(1-z) + t_1\}}, \qquad (t = [t_0 : t_1] \in \mathbb{P}^1). \tag{79}$$

Proof. In this case (77) gives the 1-parameter family of rational solutions (78). Moreover the first equation of system (1) admits a constant solution $q(z) \equiv 1$. Substituting this into the second equation yelds the Riccati equation $z(z-1)p' + (1-z)p^2 + p = 0$. Change of dependent variable $p = -zf'/f$ takes this into the linear equation $(1-z)f'' - f' = 0$, whose nontrivial solutions are given by $f = t_0 \log(1-z) + t_1$ with $(t_0, t_1) \in \mathbb{C}^2 - \{(0,0)\}$. Thus the Riccati equation has the 1-parameter family of single-valued solutions $p = p(z; t)$ as in (79). $\square$

Now we proceed to the proof of Theorem 1.3. From now on we fix the indices as $(i, j, k) = (3, 1, 2)$ in accordance with the choice of indices in §8. First we treat the fixed point case.

Lemma 9.5 *The set $\widetilde{\mathrm{Fix}}_j(\theta)$ is exhausted by meromorphic solutions around $z = 0$.*

Proof. Case-by-case check based on the "injection-implies-surjection" principle described in §1.

Example 9.6 ($\emptyset$) We combine the results of Example 7.1, Lemmas 8.1 and 8.2. A key observation is that Lemmas 8.1 and 8.2 give us as many meromorphic solutions around $z = 0$ as the cardinality of the set $\widetilde{\mathrm{Fix}}_j(\theta)$ in (49). For example, if $\widetilde{P}(b_i, b_4; b_j, b_k) \in \widetilde{\mathrm{Fix}}_j(\theta)$, then the existing and smoothness conditions for it (see Table 3) makes it possible to apply Lemma 8.2 to conclude that the meromorphic solution (66) exists corresponding to the fixed point $\widetilde{P}(b_i, b_4; b_j, b_k)$. Thus the set $\widetilde{\mathrm{Fix}}_j(\theta)$ is exhausted by meromorphic solutions around $z = 0$.

Example 9.7 ($\boldsymbol{A_1}$) We combine the results of Example 7.2, Lemmas 8.1, 8.2 and 9.1. First we notice that the two single-valued Riccati solutions in Lemma 9.1 correspond to the two Riccati fixed points $\widetilde{\mathrm{Fix}}_j^e(\theta) = \{p, q\}$ in (50). On the other hand, for the same reason as in Example 9.6, formulas (65) and (67) in Lemmas 8.1 and 8.2 give us as many meromorphic solutions around $z = 0$ as the cardinality of smooth fixed points $\widetilde{\mathrm{Fix}}_j^{\circ}(\theta) = \{\{\widetilde{P}(b_i, b_4^{-1}; b_j, b_k),\ \widetilde{P}(b_j, b_k^{-1}; b_i, b_4)\}\}$ in (50). Thus the set $\widetilde{\mathrm{Fix}}_j(\theta)$ is exhausted by meromorphic solutions around $z = 0$.

Example 9.8 ($\boldsymbol{A_2}$) We combine the results of Example 7.3, Lemmas 8.2 and 9.2. As (51) shows, the set $\widetilde{\mathrm{Fix}}_j(\theta)$ consists of the four points p_0, p_+, p_- and $\widetilde{P}(b_j, b_k^{-1}; b_i, b_4)$. On the other hand, we have the three single-valued Riccati solutions of Lemma 9.2 and one non-Riccati holomorphic solution (67). Clearly, the three Riccati solutions correspond to the points p_0, p_+ and p_-, while the non-Riccati solution corresponds to the remaining point $\widetilde{P}(b_j, b_k^{-1}; b_i, b_4)$. Thus any single-valued solution around $z = 0$ is a meromorphic solution.

Example 9.9 ($\boldsymbol{A_1^{\oplus 2}}$) We combine the results of Example 7.4, Lemmas 8.2 and 8.3. First we consider the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_1^{\oplus 2})_i$. The $\mathbb{C}$-parameter family (68) of holomorphic solutions injects into the line $\widetilde{\ell}_j^+ \simeq \mathbb{C}$ in (53), so that we have an injection $\mathbb{C} \hookrightarrow \mathbb{C}$. Since this injection is holomorphic, it must be a surjection. Thus the line $\widetilde{\ell}_j^+$ is exhausted by the family (68). Moreover we have the two holomorphic solutions (66) and (67), which do not lie in the family (68). Thus they must correspond to the points $\widetilde{P}(b_i, b_4; b_j, b_k)$ and $\widetilde{P}(b_i, b_4^{-1}; b_j, b_k)$ in (53). So on the stratum of type $(A_1^{\oplus 2})_i$ the set $\widetilde{\mathrm{Fix}}_j(\theta)$ is exhausted by meromorphic solutions around $z = 0$. Next we consider the $\widetilde{W}(D_4^{(1)})$-strata of types $(A_1^{\oplus 2})_j$ and $(A_1^{\oplus 2})_k$. On these strata the equality $\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\mathrm{Fix}}_j^e(\theta)$ in (54) implies that any single-valued solution around $z = 0$ is a Riccati and hence meromorphic solution.

Example 9.10 ($\boldsymbol{A_3}$) We combine the results of Example 7.5, Lemmas 8.2, 8.4 and 9.3. First we consider the $\widetilde{W}(D_4^{(1)})$-stratum of type $(A_1^{\oplus 2})_i$. The $\mathbb{P}^1$-family (77) of single-valued Riccati solutions exactly corresponds to the exceptional curve $e_0 \simeq \mathbb{P}^1$ in (55). So the $\mathbb{C}$-family (70) of holomorphic solutions must inject into the line $\widetilde{\ell}_j^+ \simeq \mathbb{C}$ in (55). Since this injection $\mathbb{C} \hookrightarrow \mathbb{C}$ is holomorphic, it must be a surjection. Hence $\widetilde{\ell}_j^+$ is exhausted by the family (70). Moreover there is the holomorphic solution (67), which must correspond to the point $\widetilde{P}(b_i, b_4^{-1}; b_j, b_k)$ in (55). Thus on the stratum of type $(A_1^{\oplus 2})_i$ the set $\widetilde{\mathrm{Fix}}_j(\theta)$ is exhausted by meromorphic solutions around $z = 0$. Next we consider the $\widetilde{W}(D_4^{(1)})$-strata of types $(A_1^{\oplus 2})_j$ and $(A_1^{\oplus 2})_k$. On these strata the equality $\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\mathrm{Fix}}_j^e(\theta)$ in (56) implies that any single-valued solution around $z = 0$ is a Riccati and hence meromorphic solution.

Example 9.11 ($\boldsymbol{A_1^{\oplus 3}}$) We combine the results of Example 7.6 and Lemma 8.3. As (57) shows, $\widetilde{\mathrm{Fix}}_j(\theta)$ has only one line component $\widetilde{\ell}_j^+ \simeq \mathbb{C}$. Hence the $\mathbb{C}$-family (68) of holomorphic solutions must inject into this line, so that we have an inclusion $\mathbb{C} \hookrightarrow \mathbb{C}$. Since this injection is holomorphic, it must be a surjection. Thus $\widetilde{\ell}_j^+$ is exhausted by the family (68).

Example 9.12 ($\boldsymbol{D_4}$) We combine the results of Example 7.7, Lemmas 8.5 and 9.4. The $\mathbb{P}^1$-family (78) of rational Riccati solutions corresponds to the exceptional curve e_0 in (58), while the $\mathbb{P}^1$-family (79) of single-valued Riccati solutions corresponds to the exceptional curve e_j

there. Hence $\mathbb{C}$-family (71) of holomorphic solutions, which is different from (78) and (79), must inject into the line $\widetilde{\ell}_j^+ \simeq \mathbb{C}$ in (58). Since this injection $\mathbb{C} \hookrightarrow \mathbb{C}$ is holomorphic, it must be a surjection. Thus $\widetilde{\ell}_j^+$ is exhausted by the family (71).

Example 9.13 ($\boldsymbol{A_1^{\oplus 4}}$) We combine the results of Example 7.8 and Lemma 8.6. In view of $\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\ell}_j^+ \amalg \widetilde{\ell}_j^-$, the $\mathbb{C}$-family (72) of holomorphic solutions injects into the line $\widetilde{\ell}_j^{\varepsilon} \simeq \mathbb{C}$ for some sign $\varepsilon \in \{\pm 1\}$. So we have an injection $\mathbb{C} \hookrightarrow \mathbb{C}$. Since this injection is holomorphic, it must be a surjection, so that $\widetilde{\ell}_j^{\varepsilon}$ is exhausted by the family (72). Then the other $\mathbb{C}$-family (73) of holomorphic solutions injects into the remaining line $\widetilde{\ell}_j^{-\varepsilon} \simeq \mathbb{C}$. So we have another injection $\mathbb{C} \hookrightarrow \mathbb{C}$. Since this injection is holomorphic, it must be a surjection. Hence $\widetilde{\mathrm{Fix}}_j(\theta) = \widetilde{\ell}_j^+ \amalg \widetilde{\ell}_j^-$ is exhausted by the families (72) and (73). The proof of Lemma 9.5 is now complete. $\square$

Finally we argue the periodic point case using the "injection-implies-surjection" principle.

Lemma 9.14 *For any $n > 1$ the set $\widetilde{\mathrm{Per}}_j(\theta; n)$ is exhausted by algebraic n-branch solutions around $z = 0$.*

Proof. We combine Lemmas 6.6 and 8.7. First we consider the generic case where $\kappa \in \mathcal{K} - \mathbf{Wall}$, namely, where $\theta = \mathrm{rh}(\kappa)$ is such that $\Delta(\theta) \neq 0$. In this case there is no Riccati locus and hence $\widetilde{\mathrm{Per}}_j(\theta; n) = \widetilde{\mathrm{Per}}_j^{\circ}(\theta; n)$, which is biholomorphic to the disjoint union of $\varphi(n)$ copies of $\mathbb{C}^{\times}$ by Lemma 6.6. On the other hand, by Lemma 8.7, there are $\varphi(n)$ mutually disjoint $\mathbb{C}^{\times}$-parameter families of *algebraic* n-branch solutions around $z = 0$ as in (74). Number these families from 1 to $\varphi(n)$. The first family injects into a (unique) connected component ($\simeq \mathbb{C}^{\times}$) of $\widetilde{\mathrm{Per}}_j(\theta; n)$, which we call the first component, and we have an injection $\mathbb{C}^{\times} \hookrightarrow \mathbb{C}^{\times}$. Since this injection is holomorphic, it must be a surjection and hence the first component is exhausted by the first family. Consider the second family of solutions and the corresponding second component of $\widetilde{\mathrm{Per}}_j(\theta; n)$. Notice that the second component is different from the first one, because the first component is already occupied by the first family and so it cannot contain the second family. For the same reason as above, the second component is exhausted by the second family. Since the families and the components have the same cardinality $\varphi(n)$, we can repeat this argument to conclude that $\widetilde{\mathrm{Per}}_j(\theta; n)$ is exhausted by the $\varphi(n)$ families of algebraic n-branch solutions.

Next we consider the case where $\kappa \in \mathbf{Wall}$, namely, where the Riccati part $\widetilde{\mathrm{Per}}_j^{e}(\theta; n)$ may appear. Since the lemma is trivial for the Riccati part, we have only to consider the non-Riccati part $\widetilde{\mathrm{Per}}_j^{\circ}(\theta; n)$. The argument proceeds just in the same manner as in the last paragraph, once we show that the family of solutions in (74) contains no Riccati solutions (see Remark 8.8). To see this, we consider the family $\widetilde{\mathcal{S}} \to \Theta$ of surfaces $\widetilde{\mathcal{S}}(\theta)$ parametrized by $\theta \in \Theta$ and put

$$\widetilde{\mathrm{Per}}_j^{\circ}(n) = \coprod_{\theta \in \Theta} \widetilde{\mathrm{Per}}_j^{\circ}(\theta; n), \qquad \mathcal{E} = \coprod_{\theta \in \Theta} \mathcal{E}(\theta),$$

where $\mathcal{E}(\theta)$ is the exceptional set in $\widetilde{\mathcal{S}}(\theta)$. (Precisely speaking, the parameter space Θ should be replaced by a finite covering of it to get a simultaneous minimal resolution.) Then $\widetilde{\mathrm{Per}}_j^{\circ}(n)$ and $\mathcal{E}$ are closed subsets of $\widetilde{\mathcal{S}}$ which are *disjoint* by Lemma 4.5. Now we look at the family of solutions in (74). It depends continuously on $\kappa \in \mathcal{K}$. Take any point $\kappa^* \in \mathbf{Wall}$ and let $\mathcal{K} - \mathbf{Wall} \ni \kappa \to \kappa^*$. For any $\kappa \in \mathcal{K} - \mathbf{Wall}$, the family at κ is contained in $\widetilde{\mathrm{Per}}_j^{\circ}(\theta; n)$ with

$\theta = \mathrm{rh}(\kappa)$ and hence in $\widetilde{\mathrm{Per}}_j^{\circ}(n)$. Taking the limit $\kappa \to \kappa^*$, we see that the family at κ^* is contained in $\widetilde{\mathrm{Per}}_j^{\circ}(n)$, hence in $\widetilde{\mathrm{Per}}_j^{\circ}(\theta^*; n)$ with $\theta^* = \mathrm{rh}(\kappa^*)$. Since $\widetilde{\mathrm{Per}}_j^{\circ}(\theta^*; n)$ is disjoint from $\mathcal{E}(\theta^*)$, the family at κ^* contains no Riccati solutions. Therefore the proof is complete. $\square$

Now the local statement of Theorem 1.3 around a fixed singular point, say $z = 0$, is an immediate consequence of Lemmas 9.5 and 9.14. At the same time all the finite branch solutions around $z = 0$ have been classified up to Bäcklund transformations. The global statement about algebraic solutions follows readily from the local statements around $z = 0$, 1, ∞, together with the analytic Painlevé property on $Z = \mathbb{P}^1 - \{0, 1, \infty\}$. The proof of Theorem 1.3 is complete.

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